Existence of a positive solution for a right focal discrete boundary value problem


Department of Mathematics and Computer Science, Concordia College, Moorhead, MN, 56562-0001, USA
College of Arts and Sciences, Dakota State University, Madison, SD, 57042-1799, USA
Department of Mathematics, Baylor University, Waco, TX, 76798-7328, USA

Available online: 24 Jun 2011
Existence of a positive solution for a right focal discrete boundary value problem

D.R. Anderson¹, R.I. Avery², J. Henderson³*, X. Liu³ and J.W. Lyons⁴

¹Department of Mathematics and Computer Science, Concordia College, Moorhead, MN 56562-0001, USA; ²College of Arts and Sciences, Dakota State University, Madison, SD 57042-1799, USA; ³Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA

(Received 6 November 2009; final version received 17 February 2010)

An application is made to the second-order right focal discrete boundary value problem of a recent extension of the Leggett–Williams fixed point theorem, which requires neither of the functional boundaries to be invariant. A non-trivial example is also provided.

Keywords: discrete boundary value problems; right focal; fixed point theorem; positive solution

AMS (MOS) Subject Classification: 39A10

1. Introduction

For over a decade, there has been significant interest in positive solutions and multiple positive solutions for boundary value problems for finite difference equations; see, for example [5,8,9,11,13,14,17–24].

Much of this interest has been spurred on by the applicability of a number of new fixed point theorems and multiple fixed point theorems as applied to certain discrete boundary value problems; such as the Guo–Krasnosel’skii fixed point theorem [12,15], the Leggett and Williams multiple fixed point theorem [16], fixed point theorems by Avery et al. [1–4,6] and the fixed point theorem of Ge [10].

Quite recently, Avery et al. [7] have given a topological proof and extension of the Leggett–Williams fixed point theorem which does not require either of the functionals to be invariant with regard to the concave functional boundary of a functional wedge.

In this paper, we ‘demonstrate’ a technique that takes advantage of this additional flexibility of the new fixed point theorem in establishing at least one positive solution of the right focal boundary value problem for the second-order finite difference equation,

\[ \Delta^2 u(k) + f(u(k)) = 0, \quad k \in \{0, 1, \ldots, N\}, \]  \tag{1} 

\[ u(0) = \Delta u(N + 1) = 0, \]  \tag{2} 

where \( f : \mathbb{R} \to [0, \infty) \) is continuous. In Section 2, we provide some background definitions and we state the new fixed point theorem. Then, in Section 3, we apply the fixed point
theorem to obtain a positive solution to (1) and (2). In the conclusion of Section 3, we provide a non-trivial example of our existence result.

2. Background definitions and a fixed point theorem

In this section, we state some definitions used for the remainder of this paper. In addition, we include the statement of the new fixed point whose application, in a subsequent section, will yield a solution of (1) and (2).

**Definition 2.1.** Let $E$ be a real Banach space. A nonempty closed convex set $P \subseteq E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

**Definition 2.2.** A map $\alpha$ is said to be a non-negative continuous concave functional on a cone $P$ of a real Banach space $E$ if

$$\alpha : P \to [0, \infty)$$

is continuous and

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly, we say the map $\beta$ is a non-negative continuous convex functional on a cone $P$ of a real Banach space $E$ if

$$\beta : P \to [0, \infty)$$

is continuous and

$$\beta(tx + (1 - t)y) \leq t\beta(x) + (1 - t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let $\alpha$ and $\psi$ be non-negative continuous concave functionals on $P$ and $\delta$ and $\beta$ be non-negative continuous convex functionals on $P$; then, for non-negative real numbers $a$, $b$, $c$ and $d$, we define the sets:

$$A := A(\alpha, \beta, a, d) = \{x \in P : a \leq \alpha(x) \text{ and } \beta(x) \leq d\}, \quad (3)$$

$$B := B(\alpha, \delta, \beta, a, b, d) = \{x \in A : \delta(x) \leq b\} \quad (4)$$

and

$$C := C(\alpha, \psi, \beta, a, c, d) = \{x \in A : c \leq \psi(x)\}. \quad (5)$$

We say that $A$ is a ‘functional wedge with concave functional boundary defined by the concave functional $\alpha$ and convex functional boundary defined by the convex functional $\beta’$. We say that an operator $T : A \to P$ is invariant with respect to the concave functional boundary, if $a \leq \alpha(Tx)$ for all $x \in A$, and that $T$ is invariant with respect to the convex functional boundary, if $\beta(Tx) \leq d$ for all $x \in A$. Note that $A$ is a convex set.
The following theorem [7] is the new extension of the Leggett–Williams fixed point theorem.

**Theorem 2.1.** Suppose \( P \) is a cone in a real Banach space \( E \), \( \alpha \) and \( \psi \) are non-negative continuous concave functionals on \( P \), \( \delta \) and \( \beta \) are non-negative continuous convex functionals on \( P \), and for non-negative real numbers \( a, b, c \) and \( d \), the sets \( A \), \( B \) and \( C \) are as defined in (3)–(5). Furthermore, suppose that \( A \) is a bounded subset of \( P \), that \( T : A \to P \) is completely continuous and that the following conditions hold:

- (A1) \( \{ x \in A : c < \psi(x) \text{ and } \delta(x) < b \} \neq \emptyset \) and \( \{ x \in P : \alpha(x) < a \text{ and } d < \beta(x) \} = \emptyset \);
- (A2) \( \alpha(Tx) \geq \alpha(x) \) for all \( x \in B \);
- (A3) \( \beta(Tx) \leq \beta(x) \) for all \( x \in C \) and \( \delta(Tx) > b \);
- (A4) \( \beta(Tx) \leq d \) for all \( x \in C \) and
- (A5) \( \beta(Tx) \leq d \) for all \( x \in A \) with \( \psi(Tx) < c \).

Then \( T \) has a fixed point \( x^* \in A \).

### 3. Solutions of (1) and (2)

In this section, we impose growth conditions on \( f \) such that the right focal boundary value problem for the finite difference equation, (1) and (2), has a solution as a consequence of Theorem 2.1. We note that from the non-negativity of \( f \), a solution \( u \) of (1) and (2) is both non-negative and concave on \( \{0, 1, \ldots, N + 2\} \). In our application of Theorem 2.1, we will deal with a completely continuous summation operator whose kernel is Green’s function, \( H(k, \ell) \), for

\[
-\Delta^2_y = 0 \tag{6}
\]

and satisfying (2). In particular, for \( (k, \ell) \in \{0, \ldots, N + 2\} \times \{0, \ldots, N\} \),

\[
H(k, \ell) = \frac{1}{N + 2} \begin{cases} 
k, & k \in \{0, \ldots, \ell\}, \\
\ell + 1, & k \in \{\ell + 1, \ldots, N + 2\}.
\end{cases}
\]

We observe that \( H(k, \ell) \) is non-negative, and for each fixed \( \ell \in \{0, \ldots, N\} \), \( H(k, \ell) \) is non-decreasing as a function of \( k \). In addition, it is straightforward that, for \( y, w \in \{0, \ldots, N + 2\} \) with \( y \leq w \),

\[
wH(y, \ell) \geq yH(w, \ell), \quad \ell \in \{0, \ldots, N\}. \tag{7}
\]

Next, let \( E = \{ v : \{0, \ldots, N + 2\} \to \mathbb{R} \} \) be endowed with the norm, \( ||v|| = \max_{k \in \{0, \ldots, N + 2\}} |v(k)| \), and choose the cone \( P \subseteq E \) defined by

\[
P = \{ v \in E : v \text{ is non-decreasing and non-negative-valued on } \{0, \ldots, N + 2\}, \text{ and } \Delta^2_y(k) \leq 0, k \in \{0, \ldots, N\} \}.
\]

We note that, for any \( u \in P \) and \( y, w \in \{0, \ldots, N + 2\} \) with \( y \leq w \),

\[
wu(y) \geq yu(w). \tag{8}
\]
For fixed \( \nu, \tau, \mu \in \{1, \ldots, N+2\} \), and \( \nu \in P \), we define non-negative concave functionals \( \alpha \) and \( \psi \) on \( P \) by
\[
\alpha(\nu) = \min_{k \in \{\tau, \ldots, N+2\}} \nu(k) = \nu(\tau),
\]
\[
\psi(\nu) = \min_{k \in \{\mu, \ldots, N+2\}} \nu(k) = \nu(\mu),
\]
and non-negative, convex functionals \( \delta \) and \( \beta \) on \( P \) by
\[
\delta(\nu) = \max_{k \in \{0, \ldots, \nu\}} \nu(k) = \nu(0),
\]
\[
\beta(\nu) = \max_{k \in \{0, \ldots, N+2\}} \nu(k) = \nu(N+2).
\]

Now, we put growth conditions on \( f \) such that (1) and (2) have at least one solution \( u^* \in A(\alpha, \beta, \tau d/(N+2), d) \), for a suitable choice of \( d \).

**Theorem 3.1.** Let \( \tau, \mu, \nu \in \{1, \ldots, N+2\} \), with \( 0 < \tau \leq \mu < \nu \leq N+2 \), let \( d \) and \( m \) be positive real numbers with \( 0 < m \leq (d \mu)/(N+2) \), and suppose \( f : [0, \infty) \to [0, \infty) \) is continuous and satisfies the following conditions:

(i) \( f(\nu) \geq d/(\nu - \tau) \), for \( w \in [\tau d/(N+2), \nu d/(N+2)] \),

(ii) \( f(\nu) \) is decreasing, for \( w \in [0, m] \), and \( f(m) \geq f(w) \), for \( w \in [m, d] \),

(iii) \[
\frac{\sum_{\ell=0}^{\mu} \frac{\ell + 1}{N+2} f\left(\frac{m\ell}{\mu}\right)}{\sum_{\ell=\mu+1}^{N} \frac{\ell + 1}{N+2} f(\ell)} \leq d - \frac{m}{N+2}.
\]

Then, the discrete right focal boundary value problem (1) and (2) has at least one positive solution \( u^* \in A(\alpha, \beta, \tau d/(N+2), d) \).

**Proof.** At the outset of the proof, we put
\[
a = \frac{\tau d}{N+2}, \quad b = \frac{\nu d}{N+2} \quad \text{and} \quad c = \frac{\mu d}{N+2},
\]
and let sets \( A, B, C \subseteq P \), relative to constants \( a, b, c, d \) and functionals \( \alpha, \beta, \delta, \psi \), be as defined in Section 1 by (3)–(5), respectively.

Next, we define the summation operator \( T : E \to E \) by
\[
Tu(k) = \sum_{\ell=0}^{N} H(k, \ell) f(u(\ell)), \quad u \in E, \quad k \in \{0, \ldots, N+2\}.
\]

It is immediate that \( T \) is completely continuous, and it is well known that \( u \in E \) is a solution of (1) and (2) if, and only if \( u \) is a fixed point of \( T \). We now show that the conditions of Theorem 2.1 are satisfied with respect to \( T \).

First, we note that, for each \( u \in A \), we have
\[
d \geq \beta(u) = \max_{k \in \{0, \ldots, N+2\}} u(k) = u(N+2),
\]
and we conclude that \( A \) is a bounded subset of \( P \).
Next, if we let $u \in A \subset P$, then $Tu(k) = \sum_{\ell=0}^{N} H(k, \ell) f(u(\ell)) \geq 0$ on $\{0, \ldots, N+2\}$. Moreover, $\Delta^2(Tu)(k) = -f(u(k)) \leq 0$, and so $\Delta(Tu)(k)$ is non-increasing on $\{0, \ldots, N+1\}$. From properties of $H(k, \ell)$, $\Delta(Tu)(N+1) = 0$, and so $\Delta(Tu)(k) \geq 0$ on $\{0, \ldots, N+1\}$. Thus, $(Tu)(k)$ is non-decreasing on $\{0, \ldots, N+2\}$. We conclude that $T : A \to P$.

To verify (A1) of Theorem 2.1 is satisfied, for any $K \in [2d/(2N + 3 - \mu), 2d/(2N + 3 - \nu)]$, we define

$$u_K(k) = \sum_{\ell=0}^{N} KH(k, \ell).$$

Then,

$$u_K(k) = \frac{K}{N+2} \sum_{\ell=0}^{k-1} (K(\ell + 1) + \sum_{\ell=k}^{N} Kk - \frac{Kk}{2} (2N+3-k),$$

so that

$$\alpha(u_K) = u_K(\tau) = \frac{K}{2(N+2)} (2N+3-\tau) \geq \left( \frac{2d}{2N+3-\mu} \right) \frac{(2N+3-\tau)\tau}{2(N+2)} \geq \frac{d}{N+2} = a,$$

and

$$\beta(u_K) = u_K(N+2) = \frac{K}{2(N+2)} (2N+3-(N+2)) = \frac{K}{2}(N+2)$$

$$\leq \left( \frac{2d}{2N+3-\nu} \right) \frac{(N+2)}{2} < d.$$  

In particular, $u_K \in A$. Moreover, $u_K$ has the properties,

$$\psi(u_K) = u_K(\mu) = \frac{K\mu}{2(N+2)} (2N+3-\mu) \geq \left( \frac{2d}{2N+3-\mu} \right) \frac{(2N+3-\mu)\mu}{2(N+2)} = \frac{\mu d}{N+2} = c,$$

and

$$\delta(u_K) = u_K(\nu) = \frac{K\nu}{2(N+2)} (2N+3-\nu) \leq \left( \frac{2d}{2N+3-\nu} \right) \frac{(2N+3-\nu)\nu}{2(N+2)} = \frac{\nu d}{N+2} = b,$$

so that,

$$\{u \in A : c < \psi(u) \text{ and } \delta(u) < b \} \neq \emptyset.$$  

Next, we let $u \in P$ with $\beta(u) > d$. Then, by (8),

$$\alpha(u) = u(\tau) \geq \frac{\tau}{N+2} u(N+2) = \frac{\tau \beta(u)}{N+2} > \frac{\tau d}{N+2} = a.$$
Hence,

\[ \{ u \in P : \alpha(u) < a \text{ and } d < \beta(u) \} = \emptyset. \]

Thus, part (A1) of Theorem 2.1 is satisfied.

Turning to (A2) of Theorem 2.1, we choose \( u \in B \). Then, by (i),

\[
\alpha(Tu) = \sum_{\ell=0}^{N} H(\tau, \ell) f(u(\ell)) \geq \frac{d}{\nu - \tau} \sum_{\ell=0}^{\nu} H(\tau, \ell) = \left( \frac{d}{\nu - \tau} \right) \frac{(\nu - \tau)N}{N + 2} = \frac{d}{N + 2} = a.
\]

Hence, (A2) of Theorem 2.1 is also satisfied.

Next, we establish (A3) of Theorem 2.1. To that end, we choose \( u \in A \) with \( d(Tu) \).

Then, by (7), we have

\[
\alpha(Tu) = \sum_{\ell=0}^{N} H(\tau, \ell) f(u(\ell)) \geq \frac{\tau}{\nu} \sum_{\ell=0}^{N} H(\nu, \ell) f(u(\ell)) = \frac{\tau}{\nu} \delta(Tu) > \frac{\tau}{\nu} \left( \frac{d \nu}{N + 2} \right) = a,
\]

and hence (A3) holds.

For part (A4) of Theorem 2.1, let \( u \in C \). Since \( \Delta^2 u(k) \leq 0 \), \( u(k) \) is concave, and hence, for \( \ell \in \{0, \ldots, \mu \} \), we have

\[ u(\ell) \geq \frac{c \ell}{\mu} \geq \frac{m \ell}{\mu}. \]

By conditions (ii) and (iii),

\[
\beta(Tu) = \sum_{\ell=0}^{N} H(N + 2, \ell) f(u(\ell)) = \sum_{\ell=0}^{N} \frac{\ell + 1}{N + 2} f(u(\ell))
\]

\[ = \sum_{\ell=0}^{\mu} \frac{\ell + 1}{N + 2} f(u(\ell)) + \sum_{\ell=\mu+1}^{N} \frac{\ell + 1}{N + 2} f(u(\ell)) \]

\[ \leq \sum_{\ell=0}^{\mu} \frac{\ell + 1}{N + 2} f\left( \frac{m \ell}{\mu} \right) + \sum_{\ell=\mu+1}^{N} \frac{\ell + 1}{N + 2} f(m) \]

\[ \leq d - \sum_{\ell=\mu+1}^{N} \frac{\ell + 1}{N + 2} f(m) + \sum_{\ell=\mu+1}^{N} \frac{\ell + 1}{N + 2} f(m) = d. \]

And so (A4) also holds.

For the final part, it remains to establish (A5). In that direction, let \( u \in A \) with \( \psi(Tu) < c \). This time, by (7), we have

\[
\beta(Tu) = \sum_{\ell=0}^{N} H(N + 2, \ell) f(u(\ell)) \leq \frac{N + 2}{\mu} \sum_{\ell=0}^{N} H(\mu, \ell) f(u(\ell)) = \frac{N + 2}{\mu} (Tu)(\mu) \]

\[ = \frac{N + 2}{\mu} \psi(Tu) \leq \left( \frac{N + 2}{\mu} \right) c = d, \]
and (A5) of Theorem 2.1 holds. Thus, $T$ has a fixed point $u^* \in A$, and as such, $u^*$ is a desired positive solution of (1) and (2).

The proof is complete.

Example. Let $N = 8$, $\tau = 1$, $\mu = 2$, $\nu = 10$, $d = 1$, and $m = 1/9$. Notice that $0 < \tau \leq \mu < \nu \leq 10 = N + 2$, and $0 < m = 1/9 \leq 1/5 = d\mu/(N + 2)$. With $a = 1/10$, $b = 1$ and $c = 1/5$, we define a continuous $f : [0, \infty) \to [0, \infty)$ by

$$f(w) = \begin{cases} 8w + 1, & 0 \leq w \leq \frac{1}{9}, \\ \frac{1}{9}, & w > \frac{1}{9}. \end{cases}$$

Then,

(i) $f(w) \equiv 1/9$, for $w \in [1/10, 1]$,

(ii) $f(w)$ is decreasing on $[0, 1/9]$, and $f(1/9) = f(w)$, for $w \in [1/9, 1]$ and

(iii) $\sum_{\ell=0}^{2} \frac{\ell + 1}{10} f\left(\frac{\ell}{18}\right) = \frac{22}{90} \leq \frac{51}{90} = 1 - \sum_{\ell=3}^{10} \frac{\ell + 1}{10} f\left(\frac{1}{9}\right)$.

Therefore, by Theorem 3.1, the right focal boundary value problem,

$$\Delta^2 u(k) + f(u(k)) = 0, \quad k \in \{0, \ldots, 8\},$$

$$u(0) = 0 = \Delta u(9),$$

has at least one positive solution, $u^*$, with

$$\frac{1}{10} \leq u^*(1) \quad \text{and} \quad u^*(10) \leq 1.$$

Notes
1. Email: andersod@cord.edu
2. Email: rich.avery@dsu.edu
3. Email: xueyan_liu@baylor.edu
4. Email: jeff_lyons@baylor.edu

References