**Abstract**

We study the existence of eigenvalue intervals for the even-order dynamic equation on time scales,
\[
(-1)^n x^{(Δ∇)^n} (t) = λc(t)f(x(t)), \quad t ∈ [a, b]
\]
satisfying the boundary conditions
\[
α_{i+1}x^{(Δ∇)^i}(a) − β_{i+1}x^{(Δ∇)^i}(a) = 0, \quad γ_{i+1}x^{(Δ∇)^i}(b) + δ_{i+1}x^{(Δ∇)^i}(b) = 0
\]
for 0 ≤ i ≤ n − 1, where \(f\) is a positive function and \(c\) is a nonnegative function that is allowed to vanish on some subintervals of \([a, b]\) of the time scale. The methods involve applications of Krasnoselskii’s fixed-point theorem for operators on a cone in a Banach space.

**Key words:** Green’s function, boundary value problem, time scales, cone.

**AMS Subject Classification:** 34B15, 34B10, 34B18.

**1 Introduction**

In this paper we are concerned with the existence of eigenvalues for the even-order dynamic equation on time scales
\[
(-1)^n x^{(Δ∇)^n} (t) = λc(t)f(x(t)), \quad t ∈ [a, b]
\]
satisfying Sturm-Liouville-like boundary conditions
\[
α_{i+1}x^{(Δ∇)^i}(a) − β_{i+1}x^{(Δ∇)^i}(a) = 0, \quad γ_{i+1}x^{(Δ∇)^i}(b) + δ_{i+1}x^{(Δ∇)^i}(b) = 0.
\]
Here \( n \geq 1 \) and \( 0 \leq i \leq n - 1 \), with \( a \in \mathbb{T}_\kappa^n, b \in \mathbb{T}_\kappa^n \) for a time scale \( \mathbb{T} \), and \( \sigma^n(a) < \rho^n(b) \). We take \( \alpha_j, \beta_j, \gamma_j, \delta_j \geq 0 \) and

\[
d_j := \gamma_j \beta_j + \alpha_j \delta_j + \alpha_j \gamma_j (b - a) > 0.
\]

A solution \( x \in C^{(2n)}_{ld}[a, b] \) of (1), (2) is defined on \([\rho^n(a), \sigma^n(b)]\); the solution is positive if it satisfies (1), (2), is nonnegative and is not identically zero on \([a, b] \).

There has been much interest recently in this area of obtaining optimal eigenvalue intervals of boundary value problems, often using Krasnoselskii [25] fixed point theorems to obtain intervals based on positive solutions inside a cone. A few papers along these lines are Agarwal, Bohner, and Wong [3], Anderson and Davis [5], Davis, Henderson, Prasad, and Yin [14], Eloe and Henderson [15], Erbe, Hu, and Wang [20], Erbe and Tang [21], Henderson and Wang [23], Jiang and Liu [24], Wong and Agarwal [26].

When seeking eigenintervals of boundary value problems for dynamic equations on time scales, many of these same methods carry over; see Agarwal, Bohner, and Wong [2], Anderson [4, 6], Chyan, Davis, Henderson, and Yin [11], Chyan and Henderson [12], Chyan, Henderson, and Pan [13], and Erbe and Peterson [17, 18, 19], for example.

The next two sections introduce the basic notation and concepts for time scales, and the Green’s function for the boundary value problem in question.

2 Preliminaries About Time Scales

A time scale \( \mathbb{T} \) is any nonempty closed subset of \( \mathbb{R} \). Hilger [22] initially introduced time scales with the twin goals of unifying the continuous and discrete calculus and extending the results to a dynamic calculus for general time scales. Some other early papers in this area include Agarwal and Bohner [1], Atici and Guseinov [7], Aulbach and Hilger [8], and Erbe and Hilger [16]. For an excellent introduction to the overall area of dynamic equations on time scales, see the recent texts by Bohner and Peterson [9, 10], from which we cull the following definitions. The functions \( \sigma, \rho : \mathbb{T} \rightarrow \mathbb{T} \) are jump operators given by

\[
\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}
\]

(supplemented by \( \inf \emptyset := \sup \mathbb{T} \) and \( \sup \emptyset := \inf \mathbb{T} \)). The point \( t \in \mathbb{T} \) is left-dense, left-scattered, right-dense, right-scattered if \( \rho(t) = t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t \), respectively. If \( \mathbb{T} \) has a right-scattered minimum \( m \), define \( \mathbb{T}_\kappa := \mathbb{T} - \{m\} \); otherwise, set \( \mathbb{T}_\kappa = \mathbb{T} \). If \( \mathbb{T} \) has a left-scattered maximum \( M \), define \( \mathbb{T}_\kappa := \mathbb{T} - \{M\} \); otherwise, set \( \mathbb{T}_\kappa = \mathbb{T} \).
For \( f : T \to \mathbb{R} \) and \( t \in T^\kappa \), the delta derivative [9] of \( f \) at \( t \), denoted \( f^\Delta(t) \), is the number (provided it exists) with the property that given any \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) such that

\[
|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|
\]

for all \( s \in U \). For \( T = \mathbb{R} \), we have \( f^\Delta = f' \), the usual derivative, and for \( T = \mathbb{Z} \) we have the forward difference operator, \( f^\Delta(t) = f(t + 1) - f(t) \).

For \( f : T \to \mathbb{R} \) and \( t \in T^\kappa \), the nabla derivative [7, 10] of \( f \) at \( t \), denoted \( f^\nabla(t) \), is the number (provided it exists) with the property that given any \( \varepsilon > 0 \), there is a neighborhood \( U \) of \( t \) such that

\[
|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|
\]

for all \( s \in U \). For \( T = \mathbb{R} \), we have \( f^\nabla = f' \), the usual derivative, and for \( T = \mathbb{Z} \) we have the backward difference operator, \( f^\nabla(t) = f(t) - f(t - 1) \).

A function \( f : T \to \mathbb{R} \) is left-dense continuous or ld-continuous on \( [a, b] \), denoted \( f \in C_{ld}[a, b] \), provided it is continuous at left-dense points in \( T \) and its right-sided limits exist (finite) at right-dense points in \( T \). If \( T = \mathbb{R} \), then \( f \) is ld-continuous if and only if \( f \) is continuous. If \( T = \mathbb{Z} \), then any function is ld-continuous. It is known [7, 10] that if \( f \) is ld-continuous, then there is a function \( F(t) \) such that \( F^\nabla(t) = f(t) \). In this case, we define

\[
\int_a^b f(t)\nabla t = F(b) - F(a).
\]

If \( T = \mathbb{R} \), for example, then

\[
\int_a^b f(t)\nabla t = \int_a^b f(t)dt,
\]

with the right-hand side representing the usual Riemann integral. If \( T = h\mathbb{Z} \) for some \( h > 0 \), then

\[
\int_a^b f(t)\nabla t = \begin{cases} 
\sum_{k=a/h+1}^{b/h} hf(kh) &: a < b \\
0 &: a = b \\
-\sum_{k=b/h+1}^{a/h} hf(kh) &: b < a.
\end{cases}
\]

Throughout this paper, we use the time scale interval

\[
[a, b] := \{t \in T : a \leq t \leq b\}.
\]
3 Green’s Function

Shortly we will be concerned with a completely continuous operator whose kernel is Green’s function for the related homogeneous problem

\[-1^n x^{(\Delta\nabla)^n}(t) = 0, \quad t \in [a, b]\] (3)

satisfying boundary conditions (2). For \(1 \leq j \leq n\), let \(G_j(t, s)\) be Green’s function for the boundary value problems

\[-x^{\Delta\nabla}(t) = 0, \quad t \in [a, b]\]

\[\alpha_j x(a) - \beta_j x^{\Delta}(a) = 0, \quad \gamma_j x(b) + \delta_j x^{\Delta}(b) = 0.\]

Then, for \(1 \leq j \leq n\),

\[G_j(t, s) = \begin{cases} \frac{1}{h_j} \{\alpha_j(t-a) + \beta_j\} \{\gamma_j(b-s) + \delta_j\} : & t \leq s, \\ \frac{1}{h_j} \{\alpha_j(s-a) + \beta_j\} \{\gamma_j(b-t) + \delta_j\} : & s \leq t. \end{cases}\] (4)

It is shown in [4] that for all \(s, t \in [a, b]\),

\[g_j(t)G_j(s, s) \leq G_j(t, s) \leq G_j(s, s),\] (5)

where

\[g_j(t) := \min \left\{ \frac{\alpha_j(t-a) + \beta_j}{\alpha_j(b-a) + \beta_j}, \frac{\gamma_j(b-t) + \delta_j}{\gamma_j(b-a) + \delta_j} \right\} < 1,\] (6)

for \(1 \leq j \leq n\). Let \(H_1(t, s) = G_1(t, s)\), and recursively define

\[H_j(t, s) = \int_a^b H_{j-1}(t, r)G_j(r, s)\nabla r\]

for \(2 \leq j \leq n\). Then \(H_n(t, s)\) is Green’s function for (3), (2); \(x\) is a solution of (1), (2) if and only if

\[x(t) = \lambda \int_a^b H_n(t, s)c(s)f(x(s))\nabla s\]

for a given \(\lambda\).

Example 3.1 Let \(n = 2\), \(\mathbb{T} = \mathbb{Z}\), and consider the discrete, central-difference Lidstone problem

\[x^{\Delta\nabla\Delta\nabla}(t) = 0, \quad t \in [a, b]\]

\[x(a) = x^{\Delta\nabla}(a) = 0, \quad x(b) = x^{\Delta\nabla}(b) = 0.\]
Because of the Lidstone boundary conditions,

\[ H_1(t, s) = G_j(t, s) = G(t, s) := \begin{cases} \frac{(t-a)(b-s)}{b-a} & : \ t \leq s \\ \frac{(s-a)(b-t)}{b-a} & : \ s \leq t \end{cases} \]

and

\[ H_2(t, s) = \sum_{r=a+1}^{b} G(t, r)G(r, s) = \begin{cases} u_2(t, s) : & t \leq s \\ v_2(t, s) : & s \leq t. \end{cases} \]

For \( t \leq s \),

\[
u_2(t, s) = \sum_{r=a+1}^{t} \frac{(r-a)(b-t)(r-a)(b-s)}{(b-a)^2} + \sum_{r=t+1}^{s} \frac{(t-a)(b-r)(r-a)(b-s)}{(b-a)^2} \\
+ \sum_{r=s+1}^{b} \frac{(t-a)(b-r)(s-a)(b-r)}{(b-a)^2} = \frac{(t-a)(b-s)(2bt + 1 - s^2 - t^2 - 2a(b-s))}{6(b-a)};
\]

similarly for \( t \geq s \),

\[
v_2(t, s) = \frac{(s-a)(b-t)(2bs + 1 - t^2 - s^2 - 2a(b-t))}{6(b-a)} = u_2(s, t).
\]

**Example 3.2** Let \( n = 2 \), \( T = [0, 1] \cup [2, 3] \), and again consider the Lidstone problem

\[ x^\Delta \nabla \Delta (t) = 0, \quad t \in [0, 1] \cup [2, 3] \]

\[ x(0) = x^\Delta (0) = 0, \quad x(3) = x^\Delta (3) = 0. \]

Because of the Lidstone boundary conditions,

\[ H_1(t, s) = G_j(t, s) = G(t, s) := \begin{cases} \frac{t(3-s)}{3} & : \ t \leq s \\ \frac{s(3-t)}{3} & : \ s \leq t \end{cases} \]

and

\[ H_2(t, s) = \int_{0}^{3} G(t, r)G(r, s)\nabla r = \begin{cases} u_2(t, s) : & t \leq s \\ v_2(t, s) : & s \leq t. \end{cases} \]

For \( 2 < t \leq s \leq 3 \),

\[
u_2(t, s) = \int_{0}^{t} \frac{r^2(3-t)(3-s)}{9} \nabla r + \int_{t}^{s} \frac{t(3-r)r(3-s)}{9} dr + \int_{s}^{3} \frac{t(3-r)s(3-r)}{9} dr = \frac{1}{54}(3-s)(30 - t(10 - 18s + 3s^2) - 3t^3);
\]
similarly for $3 \geq t \geq s > 2$,

$$v_2(t, s) = \frac{1}{54}(3 - t)[30 - s(10 - 18t + 3t^2) - 3s^3].$$

For the rest of the paper we have the assumptions

(A1) $c$ is a nonnegative, left-dense continuous function defined on $[a, b]$ satisfying

$$0 < \int_a^b G_n(s, s)c(s)\nabla s < \infty, \quad (7)$$

where, using (4),

$$G_n(s, s) = \frac{1}{d_n} [\alpha_n(s - a) + \beta_n] [\gamma_n(b - s) + \delta_n].$$

(A2) $f : [0, \infty) \rightarrow [0, \infty)$ is continuous such that both

$$f_0 := \lim_{x \rightarrow 0^+} \frac{f(x)}{x} \quad \text{and} \quad f_{\infty} := \lim_{x \rightarrow \infty} \frac{f(x)}{x}$$

exist.

Let $a < \xi < \omega < b$ be chosen from $\mathbb{T}$ such that

$$\int_{\xi}^{\omega} G_n(s, s)c(s)\nabla s > 0; \quad (8)$$

as $c$ is a nonnegative function, this allows $c$ to vanish on some subintervals. Following the ideas of [26] in a related difference equations case, let

$$k_j := \min_{t \in [\xi, \omega]} g_j(t) \quad (9)$$

for $g_j$ as in (6),

$$L_j := \int_a^b G_j(r, r)\nabla r, \quad 1 \leq j \leq n,$$

$$L := \prod_{j=1}^{n-1} L_j,$$

$$M_j := \int_{\xi}^{\omega} G_j(r, r)\nabla r, \quad 1 \leq j \leq n,$$
and
\[ K := k_n \prod_{j=1}^{n-1} \frac{k_j M_j}{L_j} < 1. \] (10)

Using mathematical induction it is straightforward to see that
\[ 0 \leq H_n(t, s) \leq LG_n(s, s), \quad t, s \in [a, b] \] (11)
and
\[ KLG_n(s, s) \leq H_n(t, s), \quad t \in [\xi, \omega], \quad s \in [a, b]. \] (12)

## 4 Eigenvalue Intervals

To establish eigenvalue intervals we will employ the following fixed point theorem due to Krasnoselskii [25].

**Theorem 4.1** Let $E$ be a Banach space, $P \subseteq E$ be a cone, and suppose that $\Omega_1$, $\Omega_2$ are bounded open balls of $E$ centered at the origin with $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either
\[
(i) \quad \|Au\| \leq \|u\|, \quad u \in P \cap \partial \Omega_1 \text{ and } \|Au\| \geq \|u\|, \quad u \in P \cap \partial \Omega_2, \text{ or }
\]
\[
(ii) \quad \|Au\| \geq \|u\|, \quad u \in P \cap \partial \Omega_1 \text{ and } \|Au\| \leq \|u\|, \quad u \in P \cap \partial \Omega_2
\]
holds. Then $A$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $B$ denote the Banach space $C_{Id}[a, b]$ with the norm $\|x\| = \sup_{t \in [a, b]} |x(t)|$. Define the cone $P \subset B$ by
\[ P = \{ x \in B : x(t) \geq 0 \text{ on } [a, b], \text{ and } x(t) \geq K\|x\| \text{ on } [\xi, \omega] \}, \]
where $K$ is given in (10).

**Theorem 4.2** Suppose $(A1)$ and $(A2)$ hold. Then for each $\lambda$ satisfying
\[
\frac{1}{f_\infty K^2 L \int_{\xi}^{\omega} G_n(s, s)c(s)\nabla s} < \lambda < \frac{1}{f_0 L \int_{a}^{b} G_n(s, s)c(s)\nabla s} \]
(13)
there exists at least one positive solution of (1), (2) in $P$, for $\xi, \omega$ as in (8).
Proof: Let \( \lambda \) be as in (13), and let \( \epsilon > 0 \) be such that

\[
\frac{1}{(f_\infty - \epsilon)K^2L} \int_\xi^\omega G_n(s, s)c(s)\nabla s \leq \lambda \leq \frac{1}{(f_0 + \epsilon)L} \int_a^b G_n(s, s)c(s)\nabla s.
\]

(14)

Since \( x \) is a solution of (1), (2) if and only if

\[
x(t) = \lambda \int_a^b H_n(t, s)c(s)f(x(s))\nabla s, \quad t \in [a, b],
\]

define the operator \( T : \mathcal{P} \to \mathcal{B} \) by

\[
(Tx)(t) := \lambda \int_a^b H_n(t, s)c(s)f(x(s))\nabla s, \quad x \in \mathcal{P}.
\]

(15)

We seek a fixed point of \( T \) in \( \mathcal{P} \) by establishing the hypotheses of Theorem 4.1. First, if \( x \in \mathcal{P} \) then by (5) we have

\[
(Tx)(t) = \lambda \int_a^b H_n(t, s)c(s)f(x(s))\nabla s
\leq \lambda L \int_a^b G_n(s, s)c(s)f(x(s))\nabla s,
\]

so that for \( t \in [\xi, \omega] \),

\[
(Tx)(t) = \lambda \int_a^b H_n(t, s)c(s)f(x(s))\nabla s
\geq \lambda KL \int_a^b G_n(s, s)c(s)f(x(s))\nabla s
\geq \lambda K \int_a^b H_n(s, s)c(s)f(x(s))\nabla s
\geq K \|Tx\|.
\]

Therefore \( T : \mathcal{P} \to \mathcal{P} \). Moreover, \( T \) is completely continuous by a typical application of the Ascoli-Arzela Theorem.

Now consider \( f_0 \). There exists an \( R_1 > 0 \) such that \( f(x) \leq (f_0 + \epsilon)x \) for \( 0 < x \leq R_1 \) by the definition of \( f_0 \). Pick \( x \in \mathcal{P} \) with \( \|x\| = R_1 \). Using (5) we have

\[
(Tx)(t) \leq \lambda (f_0 + \epsilon)L\|x\| \int_a^b G_n(s, s)c(s)\nabla s
\leq \|x\|
\]
from the right side of (14). As a result, \( \|Tx\| \leq \|x\| \). Thus, take
\[
\Omega_1 := \{ x \in B : \|x\| < R_1 \}
\]
so that \( \|Tx\| \leq \|x\| \) for \( x \in \mathcal{P} \cap \partial\Omega_1 \).

Next consider \( f_\infty \). Again by definition there exists an \( R'_2 > R_1 \) such that \( f(x) \geq (f_\infty - \epsilon)x \) for \( x \geq R'_2 \); take \( R_2 = \max\{2R_1, R'_2/K\} \). If \( x \in \mathcal{P} \) with \( \|x\| = R_2 \), then for \( t \in [\xi, \omega] \) we have
\[
x(t) \geq K\|x\| = KR_2.
\]
(16)
Define \( \Omega_2 := \{ x \in B : \|x\| < R_2 \} \); using (16) for \( t \in [\xi, \omega] \) we get
\[
(Tx)(t) \geq \lambda KL \int_\xi^\omega G_n(s,s)c(s)f(x(s))\nabla s
\geq \lambda KL(f_\infty - \epsilon) \int_\xi^\omega G_n(s,s)c(s)x(s)\nabla s
\geq \lambda (f_\infty - \epsilon)K^2R_2L \int_\xi^\omega G_n(s,s)c(s)\nabla s
\geq R_2
= \|x\|,
\]
where we have used the left side of (14). Hence we have shown that
\[
\|Tx\| \geq \|x\|, \quad x \in \mathcal{P} \cap \partial\Omega_2.
\]
An application of Theorem 4.1 yields the conclusion of the theorem.

**Theorem 4.3** Suppose (A1) and (A2) hold. Then for each \( \lambda \) satisfying
\[
\frac{1}{f_0K^2L \int_\xi^\omega G_n(s,s)c(s)\nabla s} \leq \lambda \leq \frac{1}{f_\infty L \int_a^b G_n(s,s)c(s)\nabla s}
\]
(17)
there exists at least one positive solution of (1), (2) in \( \mathcal{P} \).

**Proof:** Let \( \lambda \) be as in (17) and let \( \eta > 0 \) be such that
\[
\frac{1}{(f_0 - \eta)K^2L \int_\xi^\omega G_n(s,s)c(s)\nabla s} \leq \lambda \leq \frac{1}{(f_\infty + \eta)L \int_a^b G_n(s,s)c(s)\nabla s}.
\]
(18)
Let \( T \) be the completely continuous, cone-preserving operator defined in (15). We seek a fixed point of \( T \) in \( \mathcal{P} \) by establishing the hypotheses of Theorem 4.1.
First, consider $f_0$. There exists an $R_1 > 0$ such that $f(x) \geq (f_0 - \eta)x$ for $0 < x \leq R_1$ by the definition of $f_0$. Pick $x \in P$ with $\|x\| = R_1$. For $t \in [\xi, \omega]$, where $\xi, \omega$ are as in (8), we have

$$x(t) \geq K\|x\| = KR_1.$$  \hspace{1cm} (19)

Using the left side of (18) and (19) we get, for $t \in [\xi, \omega]$,

$$(Tx)(t) \geq \lambda KL \int_\xi^\omega G_n(s, s)c(s)f(x(s))\nabla s \geq \lambda(f_0 - \eta)KL \int_\xi^\omega G_n(s, s)c(s)x(s)\nabla s \geq \lambda(f_0 - \eta)R_1K^2L \int_\xi^\omega G_n(s, s)c(s)\nabla s \geq R_1 \geq \|x\|.$$  

Therefore $\|Tx\| \geq \|x\|$. This prompts us to define

$$\Omega_1 := \{x \in \mathcal{B} : \|x\| < R_1\},$$

whereby our work above confirms

$$\|Tx\| \geq \|x\|, \ x \in P \cap \partial\Omega_1.$$  

Next consider $f_\infty$. Again by definition there exists an $R_2' > R_1$ such that $f(x) \leq (f_\infty + \eta)x$ for $x \geq R_2'$. If $f$ is bounded, there exists $M > 0$ with $f(x) \leq M$ for all $x \in (0, \infty)$. Let

$$R_2 := \max\{2R_2', \lambda LM \int_a^b G_n(s, s)c(s)\nabla s\}.$$  

If $x \in P$ with $\|x\| = R_2$, then we have

$$(Tx)(t) = \lambda \int_a^b H_n(t, s)c(s)f(x(s))\nabla s \leq \lambda L \int_a^b G_n(s, s)c(s)f(x(s))\nabla s \leq \lambda LM \int_a^b G_n(s, s)c(s)\nabla s \leq R_2 \leq \|x\|.$$
As a result, \( \|Tx\| \leq \|x\| \). Thus, take
\[
\Omega_2 := \{ x \in \mathcal{B} : \|x\| < R_2 \}
\]
so that \( \|Tx\| \leq \|x\| \) for \( x \in \mathcal{P} \cap \partial \Omega_2 \). If \( f \) is unbounded, take \( R_2 := \max\{2R_1, R'_2\} \) such that \( f(x) \leq f(R_2) \) for \( 0 < x \leq R_2 \). If \( x \in \mathcal{P} \) with \( \|x\| = R_2 \), then we have
\[
(Tx)(t) \leq \lambda L \int_a^b G_n(s, s) c(s) f(R_2) \nabla s \\
\leq \lambda (f_\infty + \eta) R_2 L \int_a^b G_n(s, s) c(s) \nabla s \\
\leq R_2 \\
= \|x\|,
\]
where we have used the left side of (18). Hence we have shown that
\[
\|Tx\| \leq \|x\|, \quad x \in \mathcal{P} \cap \partial \Omega_2
\]
if we take
\[
\Omega_2 := \{ x \in \mathcal{B} : \|x\| < R_2 \}.
\]
Once again an application of Theorem 4.1 yields the conclusion of the theorem.

\[\square\]

**Corollary 4.4** Suppose (A1) and (A2) hold. If \( f \) is sublinear (i.e., \( f_0 = \infty \) and \( f_\infty = 0 \)), or if \( f \) is superlinear (i.e., \( f_0 = 0 \) and \( f_\infty = \infty \)), then for any \( \lambda > 0 \) the boundary value problem (1), (2) has at least one positive solution in \( \mathcal{P} \).

**Proof:** For the superlinear claim, use (13) of Theorem 4.2; for the sublinear claim, use (17) of Theorem 4.3.

\[\square\]

**References**


