KAMENEV-TYPE OSCILLATION CRITERIA FOR LINEAR HAMILTONIAN SYSTEMS

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Abstract. New oscillation criteria of Kamenev type are established for linear Hamiltonian matrix systems using generalized Riccati and integral averaging techniques. The results generalize some recent work on linear Hamiltonian systems and the related self-adjoint second-order matrix equation, without controllability or differentiability requirements on the coefficient matrix functions.

1. Introduction

We are concerned with the linear Hamiltonian system

\begin{align*}
X' &= A(t)X + B(t)Y \\
Y' &= C(t)X - A^*(t)Y, \quad t \geq t_0,
\end{align*}

(1.1)

where \(A\), \(B\), and \(C\) are continuous \(n \times n\) real-valued matrix functions such that \(B\) and \(C\) are symmetric with \(B\) positive definite. Here \(A^*\) denotes the transpose of \(A\). For any two solutions \((X_1, Y_1)\) and \((X_2, Y_2)\) of (1.1), the Wronskian \(X_1^*(t)Y_2(t) - Y_1^*(t)X_2(t)\) is a constant matrix. In particular, every solution \((X, Y)\) of (1.1) satisfies

\[X^*(t)Y(t) - Y^*(t)X(t) \equiv \text{constant}.\]

**Definition 1.1.** A solution \((X, Y)\) of (1.1) is nontrivial if \(\det X(t) \neq 0\) for at least one \(t \geq t_0\). A nontrivial solution \((X, Y)\) of (1.1) is prepared if \(X^*(t)Y(t) - Y^*(t)X(t) \equiv 0\) for all \(t \geq t_0\). A nontrivial prepared solution \((X, Y)\) of (1.1) is oscillatory if \(\det X(t)\) has arbitrarily large zeros. From the Sturm separation theorem, all nontrivial prepared solutions of (1.1) are either oscillatory or nonoscillatory; thus, system (1.1) itself is either oscillatory or nonoscillatory.

2. Main Result

**Remark 2.1.** In the theorem to follow, we employ a standard class of functions as described here. Let \(D_0 = \{(t, s) : t > s \geq t_0\}\) and \(D = \{(t, s) : t \geq s \geq t_0\}\). Let \(H \in C(D, \mathbb{R})\), \(h \in C(D_0, \mathbb{R})\), and \(k \in C([t_0, \infty), (0, \infty))\) satisfy the following three conditions:

\[(I)\quad H(t, t) = 0 \text{ for } t \geq t_0 \text{ and } H(t, s) > 0 \text{ on } D_0;\]

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(II) $H$ has a continuous and nonpositive partial derivative on $D_0$ with respect to its second variable;

(III) for all $(t, s) \in D_0$,
\[
-\frac{\partial}{\partial s} (H(t, s)k(s)) = h(t, s)\sqrt{H(t, s)k(s)}.
\] (2.1)

Some functions $H$ in this class include $(t - s)^r$ and $(\ln \frac{t}{s})^r$ for constant $r > 1$, or $\rho(t - s)$ for $\rho \in C([0, \infty), \mathbb{R})$ with $\rho(0) = 0$, $\rho(u) > 0$ and $\rho'(u) \geq 0$ for $u > 0$.

**Remark 2.2.** To simplify some of the longer expressions to follow, we introduce the following, where $f \in C^1([t_0, \infty), \mathbb{R})$.

\[
a(s) = \exp \left( -2 \int_s^t f(\tau)d\tau \right),
\]
\[
D(s) = -a(s) \left( C(s) + f'(s)I + A^*(s)B^{-1}(s)A(s) \right),
\]
\[
E(s) = \frac{1}{2} a(s) \left( B^{-1}(s)A(s) + A^*(s)B^{-1}(s) - 2f(s)I \right),
\]
\[
\Phi(s) = D(s) - 2f(s)E(s),
\]
\[
F(t, s) = -a(s) \left( \frac{1}{2} h(t, s) + f(s)\sqrt{H(t, s)k(s)} \right)^2 B^{-1}(s)
\]
\[
- h(t, s)\sqrt{H(t, s)k(s)}E(s).
\]

Note that since $B$ and $C$ are symmetric, $D$, $E$, $\Phi$, and $F$ are all symmetric as well; in particular, $H(t, s)k(s)\Phi(s) + F(t, s)$ is symmetric and has real eigenvalues.

If $A(t) \equiv 0$ in (1.1), the Hamiltonian system reduces to the second-order self-adjoint matrix differential equation
\[
(P(t)X)' + Q(t)X = 0, \quad t \geq t_0,
\] (2.2)
with $P = B^{-1}$ and $Q = -C$. In this note we will not require that $B^{-1}$ be differentiable or that $B$ satisfies any controllability requirements.

Many oscillation criteria for (1.1), (2.2) and assorted special cases have been obtained over the years; see [1]-[7], [9]. In the most recent papers, [8] uses the techniques employed here with $A \equiv 0$, while [10] allows $A \neq 0$ but takes $f \equiv 0$ and thus $a \equiv 1$, as well as $k \equiv 1$.

**Theorem 2.1.** Let the scalar functions $H$, $h$, and $k$ satisfy conditions (I) – (III) in Remark 2.1, and let the scalar function $a$ and the symmetric matrix functions $D$, $E$, $\Phi$, and $F$ be as in Remark 2.2. If there exists a function $f \in C^1([t_0, \infty), \mathbb{R})$ such that
\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \lambda_1 \left[ \int_{t_0}^t (H(t, s)k(s)\Phi(s) + F(t, s))ds \right] = \infty,
\] (2.3)
where $\lambda_1[\cdot]$ denotes the largest eigenvalue of the enclosed matrix, then (1.1) is oscillatory.
Proof. Assume rather that there exists a nontrivial prepared solution \((X, Y)\) of (1.1) such that \(X\) is nonsingular for all sufficiently large \(t\); without loss of generality, suppose that \(\det X(t) \neq 0\) for all \(t \geq t_0\). Define the generalized Riccati substitution
\[
W(t) = a(t) \left[ Y(t)X^{-1}(t) + f(t)I \right], \quad t \geq t_0.
\]
Then \(W\) is symmetric and satisfies the Riccati equation
\[
W'(t) = a(t) \left[ C(t) + f'(t)I - f^2(t)B(t) + f(t)A^*(t) + f(t)A(t) \right]
- \frac{1}{a(t)} W(t)B(t)W(t) - A^*(t)W(t) - W(t)A(t)
- f(t) \left[ 2W(t) - (W(t)B(t) + B(t)W(t)) \right].
\]
Introduce another function
\[
V(t) = W(t) + a(t)(B^{-1}(t)A(t) - f(t)I).
\]
Then, suppressing the argument, we have
\[
\frac{1}{a} V^*BV = \frac{1}{a} WBW + WA - fWB + A^*W - fBW
+ aA^*B^{-1}A - afA^* - afA + af^2B;
\]
using Remark 2.2 this simplifies to
\[
\frac{1}{a} V^*BV = -W' - 2fW - D.
\]
Since \(B\) is positive definite and symmetric, \(R := B^{1/2}\) is well defined and symmetric. Let
\[
Q(t, s) = \sqrt{\frac{H(t, s)k(s)}{a(s)}} R(s)V(s) + \frac{\sqrt{a(s)}}{2} h(t, s)R^{-1}(s)
+ f(s)\sqrt{a(s)H(t, s)k(s)}R^{-1}(s).
\]
Again suppressing the arguments, we have
\[
Q^*Q = Hk(-W' - 2fW - D) + \sqrt{Hk} \left( \frac{h}{2} + f\sqrt{Hk} \right) (V^* + V)
+ a \left( \frac{h}{2} + f\sqrt{Hk} \right)^2 B^{-1}
= Hk(-W' - D + 2fE) + h\sqrt{Hk}(W + E) + a \left( \frac{h}{2} + f\sqrt{Hk} \right)^2 B^{-1}
= -\frac{\partial}{\partial s} [(Hk)W] + Hk(-D + 2fE) + h\sqrt{Hk}E
+ a \left( \frac{h}{2} + f\sqrt{Hk} \right)^2 B^{-1}.
\]
Switching sides and using the shorthand from Remark 2.2 leads to

\[ H(t, s)k(s)\Phi(s) + F(t, s) = -\frac{\partial}{\partial s}((H(t, s)k(s))W(s)) - Q^*(t, s)Q(t, s). \]

Integrating this from \( t_0 \) to \( t \) and using the properties of \( H \) yields

\[
\int_{t_0}^{t} [H(t, s)k(s)\Phi(s) + F(t, s)] \, ds = H(t, t_0)k(t_0)W(t_0) - \int_{t_0}^{t} Q^*(t, s)Q(t, s) \, ds
\]

\[ \leq H(t, t_0)k(t_0)W(t_0). \]

We proceed as in [8]. Since \( k(t_0) > 0 \) and \( H(t, t_0) > 0 \) for \( t > t_0 \), for all \( t > t_0 \) we have that

\[
\lambda_1 \left[ \int_{t_0}^{t} [H(t, s)k(s)\Phi(s) + F(t, s)] \, ds \right] \leq \lambda_1[H(t, t_0)k(t_0)W(t_0)]
\]

\[ \leq H(t, t_0)k(t_0)\lambda_1[W(t_0)]. \]

Consequently,

\[
\frac{1}{H(t, t_0)} \lambda_1 \left[ \int_{t_0}^{t} [H(t, s)k(s)\Phi(s) + F(t, s)] \, ds \right] \leq k(t_0)\lambda_1[W(t_0)].
\]

Since the right-hand side of the inequality is constant, the left-hand side is bounded, a contradiction of (2.3). Thus (1.1) must be oscillatory. \( \square \)

With a modification of the hypotheses of Theorem 2.1, we also have the following result.

**Theorem 2.2.** Let the scalar functions \( H, h, \) and \( k \) satisfy conditions (I) – (III) in Remark 2.1, and let the scalar function \( a \) and the symmetric matrix functions \( D, E, \Phi, \) and \( F \) be as in Remark 2.2. If there exists a function \( f \in C^1([t_0, \infty), \mathbb{R}) \) such that

\[
\limsup_{t \to \infty} \frac{1}{H(t, t_0)} \left[ \int_{t_0}^{t} (H(t, s)k(s)\text{tr}\Phi(s) + \text{tr}F(t, s)) \, ds \right] = \infty,
\]

then (1.1) is oscillatory.

**References**


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