Best constant for Hyers–Ulam stability of two step sizes linear difference equations

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\textbf{Abstract}

This study deals with the Hyers–Ulam stability (HUS) for the first-order linear difference equations with two alternating step sizes, where the coefficient is allowed to be complex valued. In particular, it turns out that the best HUS constant can be determined by finding an explicit solution to the corresponding inhomogeneous linear equation. Special cases of these results validate previous literature in the field.

\section{Introduction}

In this paper, we denote the set of integers, real numbers, and complex numbers by $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Let $\sigma, \tau > 0$ be two step sizes, and let the time scale that they constitute be denoted by

$$T_{\sigma,\tau} := \{\cdots, -(\sigma + \tau) - \tau, -(\sigma + \tau), -\tau, 0, \sigma, \sigma + \tau, (\sigma + \tau) + \sigma, 2(\sigma + \tau), 2(\sigma + \tau) + \sigma, \cdots \}.$$ 

Define the difference operator $\Delta$ via

$$z^{\Delta}(t) := \begin{cases} 
\frac{z(t + \sigma) - z(t)}{\sigma} & : \frac{t}{\sigma + \tau} \in \mathbb{Z}, \\
\frac{z(t + \tau) - z(t)}{\tau} & : \frac{t - \sigma}{\sigma + \tau} \in \mathbb{Z}
\end{cases}$$

for $t \in T_{\sigma,\tau}$. We consider the first-order linear difference equation

$$z^{\Delta}(t) - \omega z(t) = 0,$$

(1.1)
where $\omega \in \mathbb{C}$ satisfies $(1 + \omega \sigma)(1 + \omega \tau) \neq 0$. Let $\mathbb{T} := \mathbb{T}_{\sigma,\tau} \cap (a, b)$, where $-\infty \leq a$ and $b \leq \infty$. Let $\mathbb{T}^\dagger := \mathbb{T} \setminus \{\max \mathbb{T}\}$ if $\max \mathbb{T}$ exists; $\mathbb{T}^\dagger := \mathbb{T}$ otherwise. The purpose of this study is to consider the Hyers–Ulam stability (HUS) for (1.1). This research area has been receiving attention in recent years since Ulam’s proposal [24] in the field of functional equations. For the Hyers–Ulam stability of functional equations and their historical background, refer to [10,11]. In 2005, Popa [19,20] started to study the Hyers–Ulam stability of the difference equation. Later, the number of studies on the Hyers–Ulam stability of difference equations has increased and is growing. For example, see [4–8,13–18,21]. It is well known that difference equations and differential equations are known as one of the dynamic equations on time scales [9]. In recent years, the studies of Hyers–Ulam stability for dynamic equations on time scales have also been developed. See [1–3,12,22,23] for results on the dynamic equations on time scales.

Now we will give the definition of Hyers–Ulam stability for (1.1).

**Definition 1.1.** Equation (1.1) has Hyers–Ulam stability (HUS) if and only if there exists a constant $K > 0$ with the following property:

For an arbitrary $\varepsilon > 0$, if a function $\zeta : \mathbb{T} \to \mathbb{C}$ satisfies

$$|\zeta(t) - \zeta(t)| \leq K\varepsilon$$

(1.2)

for all $t \in \mathbb{T}^\dagger$, then there exists a solution $z : \mathbb{T} \to \mathbb{C}$ of (1.1) such that

$$|\zeta(t) - z(t)| \leq K\varepsilon$$

for all $t \in \mathbb{T}$.

Such a constant $K$ is called an HUS constant for (1.1) on $\mathbb{T}$.

When $\sigma = \tau = h > 0$, (1.1) is reduced to the $h$-difference equation

$$\Delta_h z(t) - \omega z(t) = 0, \quad \Delta_h z(t) := \frac{z(t+h) - z(t)}{h}$$

(1.3)

and $\mathbb{T}_{\sigma,\tau} = h\mathbb{Z} := \{hk | k \in \mathbb{Z}\}$. Moreover, if $h = 1$, then, (1.3) is $z(t+1) - (1 + \omega)z(t) = 0$. Recently, the authors [4] investigated the Hyers–Ulam stability of $h$-difference equation (1.3) and its best (minimal) HUS constant. Finding an explicit best HUS constant is a very important problem because the error between the approximate solution $\zeta$ and the exact solution $z$ can be quantified by it. Baias and Popa [7,8] studied the simple case $h = 1$, and Onitsuka [17,18] treated the real-valued coefficient case $\omega \in \mathbb{R}$. For results on the best constants relevant to this study, refer to [2,5,6].

Hereafter, we assume that $\sigma \neq \tau$, so we call (1.1) a difference equation with two step sizes. In 2019, the authors [3] discussed the Hyers–Ulam stability and the best (minimal) HUS constant for (1.1) with $\omega \in \mathbb{R}$. The main purpose of this study is to find an explicit best HUS constant for the complex-valued coefficient case. The proof is presented in a completely different way from previous study [3], based on the ideas considered in [4,5].

The paper will proceed as follows. In Section 2, we establish the explicit solution for inhomogeneous difference equations with two step sizes. In Section 3, we will provide necessary, technical lemmas for future use. In Section 4, we prove our main results on HUS and provide the best HUS constant for important cases. In Section 5, we visualize some of the results by introducing polar coordinates for the complex coefficient $\omega$ in (1.1).
2. Explicit solution for inhomogeneous difference equations with two step sizes

Now we define the discrete exponential function $e_\omega$ by

$$e_\omega(t) := \begin{cases} \frac{t}{\sigma + \tau} \in \mathbb{Z}, \\ \frac{t - \sigma}{\sigma + \tau} \in \mathbb{Z} \end{cases}$$

for $t \in T_{\sigma, \tau}$ and $\omega \in \mathbb{C} \setminus \{-\frac{1}{\sigma}, -\frac{1}{\tau}\}$. Actually, we have

$$(e_\omega(t))^\Delta = \begin{cases} \frac{t}{\sigma + \tau} \in \mathbb{Z}, \\ \frac{t - \sigma}{\sigma + \tau} \in \mathbb{Z} \end{cases}$$

so that $e_\omega(t)$ is the solution to (1.1) with $e_\omega(0) = 1$. Let $t_0 \in T_{\sigma, \tau}$ and $z_0 \in \mathbb{C}$. Then,

$$z(t) = z_0 \frac{e_\omega(t)}{e_\omega(t_0)}$$

is the solution to (1.1) with the initial condition $z(t_0) = z_0$.

The purpose of this section is to find an explicit solution to the inhomogeneous difference equation with two step sizes

$$\zeta^\Delta(t) - \omega \zeta(t) = f(t)$$

for all $t \in T_{\sigma, \tau}$. Define

$$\nu(t) := \begin{cases} \frac{2t}{\sigma + \tau} \in \mathbb{Z}, \\ \frac{2t - \sigma}{\sigma + \tau} + 1 \in \mathbb{Z} \end{cases}$$

Then, the values of $\nu(t)$ are shown in the table below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\cdots$</th>
<th>$-(\sigma + \tau)$</th>
<th>$-\tau$</th>
<th>0</th>
<th>$\sigma$</th>
<th>$\sigma + \tau$</th>
<th>$(\sigma + \tau) + \sigma$</th>
<th>$2(\sigma + \tau)$</th>
<th>$2(\sigma + \tau) + \sigma$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu(t)$</td>
<td>$\cdots$</td>
<td>$-2$</td>
<td>$-1$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>$\cdots$</td>
<td></td>
</tr>
</tbody>
</table>

Hence, $\nu(t) \in \mathbb{Z}$. The following fact is established.

**Lemma 2.1.** Let $\omega \in \mathbb{C}$ satisfy $(1 + \omega \sigma)(1 + \omega \tau) \neq 0$, and define

$$\gamma(k) := \frac{1}{2} \left[ \left( \frac{\tau}{\sqrt{(1 + \omega \sigma)(1 + \omega \tau)}} + \frac{\sigma}{1 + \omega \sigma} \right) \left( \frac{1}{\sqrt{(1 + \omega \sigma)(1 + \omega \tau)}} \right)^k - \left( \frac{\tau}{\sqrt{(1 + \omega \sigma)(1 + \omega \tau)}} - \frac{\sigma}{1 + \omega \sigma} \right) \left( -\frac{1}{\sqrt{(1 + \omega \sigma)(1 + \omega \tau)}} \right)^k \right]$$

for $k \in \mathbb{Z}$. For $t_0 \in T_{\sigma, \tau}$, define
\[ S(t) := \begin{cases} 
\sum_{k=\nu(t_0)}^{\nu(t)-1} \gamma(k) f(\nu^{-1}(k)) & : t \geq t_0, \ t \in \mathbb{T}_{\sigma,\tau}, \\
- \sum_{k=\nu(t)}^{\nu(t_0)-1} \gamma(k) f(\nu^{-1}(k)) & : t \leq t_0, \ t \in \mathbb{T}_{\sigma,\tau}, 
\end{cases} \tag{2.5} \]

where \( \nu \) is given in (2.3) and \( \nu^{-1} \) is the inverse function of \( \nu \). Then, \( e_\omega(t)S(t) \) is a solution of (2.2) for all \( t \in \mathbb{T}_{\sigma,\tau} \).

**Proof.** Note that

\[ \sum_{k=\nu(t_0)}^{\nu(t)-1} \gamma(k) f(\nu^{-1}(k)) = 0, \]

which is the standard assumption. First, we will prove that

\[ Z_1(t) = e_\omega(t) \sum_{k=\nu(t_0)}^{\nu(t)-1} \gamma(k) f(\nu^{-1}(k)), \]

is a solution of (2.2) for \( t \geq t_0 \) and \( t \in \mathbb{T}_{\sigma,\tau} \). Consider the case \( \frac{t}{\sigma+\tau} \in \mathbb{Z} \). From (2.3), \( \nu(t) \) is an even integer, and we have

\[ \gamma(\nu(t)) = \frac{1}{2} \left[ \left( \frac{\tau}{\sqrt{(1+\omega\sigma)(1+\omega \tau)}} + \frac{\sigma}{1+\omega \sigma} \right) \left( \frac{1}{(1+\omega \sigma)(1+\omega \tau)} \right)^{\frac{1}{\sigma+\tau}} - \left( \frac{\tau}{\sqrt{(1+\omega\sigma)(1+\omega \tau)}} - \frac{\sigma}{1+\omega \sigma} \right) \left( \frac{1}{(1+\omega \sigma)(1+\omega \tau)} \right)^{\frac{1}{\sigma+\tau}} \right], \]

\[ = \frac{\sigma}{(1+\omega \sigma)(1+\omega \tau)} = e_\omega(t+\sigma). \tag{2.6} \]

Using this, we obtain

\[ Z_1^\Delta(t) = \frac{1}{\sigma} \left[ e_\omega(t+\sigma) \sum_{k=\nu(t_0)}^{\nu(t+\sigma)-1} \gamma(k) f(\nu^{-1}(k)) - e_\omega(t) \sum_{k=\nu(t_0)}^{\nu(t)-1} \gamma(k) f(\nu^{-1}(k)) \right] \]

\[ = \frac{1}{\sigma} \left[ e_\omega(t+\sigma) \sum_{k=\nu(t_0)}^{\nu(t)} \gamma(k) f(\nu^{-1}(k)) - e_\omega(t+\sigma) \sum_{k=\nu(t_0)}^{\nu(t)-1} \gamma(k) f(\nu^{-1}(k)) + e_\omega(t+\sigma) \sum_{k=\nu(t_0)}^{\nu(t)-1} \gamma(k) f(\nu^{-1}(k)) \right] \]

\[ = \frac{1}{\sigma} \left[ e_\omega(t+\sigma) \gamma(\nu(t)) f(t) + (e_\omega(t+\sigma) - e_\omega(t)) \sum_{k=\nu(t_0)}^{\nu(t)-1} \gamma(k) f(\nu^{-1}(k)) \right] \]

\[ = f(t) + (e_\omega(t))^\Delta \sum_{k=\nu(t_0)}^{\nu(t)-1} \gamma(k) f(\nu^{-1}(k)) = f(t) + \omega Z_1(t), \]
so that (2.2) is satisfied when \( \frac{t}{\sigma + \tau} \in \mathbb{Z} \). Next, we consider the case \( \frac{t - \sigma}{\sigma + \tau} \in \mathbb{Z} \). From (2.3), \( \nu(t) \) is an odd integer, and we find

\[
\gamma(\nu(t)) = \frac{1}{2} \left[ \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} + \frac{\sigma}{1 + \omega\sigma} \right] \left( \frac{1}{(1 + \omega\sigma)(1 + \omega\tau)} \right)^{\frac{t - \sigma}{\sigma + \tau}} \frac{1}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} + \left( \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} - \frac{\sigma}{1 + \omega\sigma} \right) \left( \frac{1}{(1 + \omega\sigma)(1 + \omega\tau)} \right)^{\frac{t - \sigma}{\sigma + \tau}} \frac{1}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} \right]
\]

\[
= \frac{\tau}{(1 + \omega\sigma)(1 + \omega\tau)} = e_\omega(t + \tau).
\]

Using this, we obtain

\[
Z_1^\Delta(t) = \frac{1}{\tau} \left[ e_\omega(t + \tau) \sum_{k = \nu(t_0)}^{\nu(t + 1) - 1} \gamma(k)f(\nu^{-1}(k)) - e_\omega(t) \sum_{k = \nu(t_0)}^{\nu(t) - 1} \gamma(k)f(\nu^{-1}(k)) \right]
\]

\[
= \frac{1}{\tau} \left[ e_\omega(t + \tau)\gamma(\nu(t))f(t) + (e_\omega(t + \tau) - e_\omega(t)) \sum_{k = \nu(t_0)}^{\nu(t) - 1} \gamma(k)f(\nu^{-1}(k)) \right]
\]

\[
= f(t) + (e_\omega(t))\Delta \sum_{k = \nu(t_0)}^{\nu(t) - 1} \gamma(k)f(\nu^{-1}(k)) = f(t) + \omega Z_1(t).
\]

Hence, \( Z_1(t) \) solves (2.2) for all \( t \geq t_0 \) and \( t \in \mathbb{T}_{\sigma, \tau} \).

Next, we will show that

\[
Z_2(t) = -e_\omega(t) \sum_{k = \nu(t)}^{\nu(t_0) - 1} \gamma(k)f(\nu^{-1}(k)),
\]

is a solution of (2.2) for \( t \leq t_0 \) and \( t \in \mathbb{T}_{\sigma, \tau} \). Again, we first consider the case \( \frac{t}{\sigma + \tau} \in \mathbb{Z} \). By (2.3), \( \nu(t) \) is an even integer, and (2.6) holds. From this, we have

\[
Z_2^\Delta(t) = -\frac{1}{\sigma} \left[ e_\omega(t + \sigma) \sum_{k = \nu(t+1)}^{\nu(t_0)} \gamma(k)f(\nu^{-1}(k)) - e_\omega(t) \sum_{k = \nu(t)}^{\nu(t_0) - 1} \gamma(k)f(\nu^{-1}(k)) \right]
\]

\[
= -\frac{1}{\sigma} \left[ e_\omega(t + \sigma) \sum_{k = \nu(t+1)}^{\nu(t_0)} \gamma(k)f(\nu^{-1}(k)) - e_\omega(t + \sigma) \sum_{k = \nu(t)}^{\nu(t_0) - 1} \gamma(k)f(\nu^{-1}(k)) + e_\omega(t + \sigma) \sum_{k = \nu(t)}^{\nu(t_0) - 1} \gamma(k)f(\nu^{-1}(k)) - e_\omega(t) \sum_{k = \nu(t)}^{\nu(t_0) - 1} \gamma(k)f(\nu^{-1}(k)) \right]
\]

\[
= -\frac{1}{\sigma} \left[ -e_\omega(t + \sigma)\gamma(\nu(t))f(t) + (e_\omega(t + \sigma) - e_\omega(t)) \sum_{k = \nu(t)}^{\nu(t_0) - 1} \gamma(k)f(\nu^{-1}(k)) \right]
\]

\[
= f(t) - (e_\omega(t))\Delta \sum_{k = \nu(t)}^{\nu(t_0) - 1} \gamma(k)f(\nu^{-1}(k)) = f(t) + \omega Z_2(t).
\]
Therefore, (2.2) is satisfied when \( \frac{t}{\sigma + \tau} \in \mathbb{Z} \). Last, consider the case \( \frac{t-\sigma}{\sigma + \tau} \in \mathbb{Z} \). By (2.3), \( \nu(t) \) is an odd integer, and (2.7) holds. Using this, we get

\[
Z_2^\Delta(t) = -\frac{1}{\tau} \left[ e_\omega(t + \tau) \sum_{k=\nu(t+\tau)}^{\nu(t_0)-1} \gamma(k)f(\nu^{-1}(k)) - e_\omega(t) \sum_{k=\nu(t)}^{\nu(t_0)-1} \gamma(k)f(\nu^{-1}(k)) \right],
\]

\[
= -\frac{1}{\tau} \left[ - e_\omega(t + \tau)\gamma(\nu(t))f(t) + (e_\omega(t + \tau) - e_\omega(t)) \sum_{k=\nu(t)}^{\nu(t_0)-1} \gamma(k)f(\nu^{-1}(k)) \right],
\]

\[
= f(t) - (e_\omega(t))^{\Delta} \sum_{k=\nu(t)}^{\nu(t_0)-1} \gamma(k)f(\nu^{-1}(k)) = f(t) + \omega Z_2(t),
\]

so that \( Z_2(t) \) solves (2.2) for all \( t \leq t_0 \) and \( t \in T_{\sigma, \tau} \). Consequently, we conclude that \( e_\omega(t)S(t) \) is a solution of (2.2) for all \( t \in T_{\sigma, \tau} \).

**Lemma 2.2.** Let \( t_0 \in T_{\sigma, \tau} \) and \( \zeta_0 \in \mathbb{C} \), and let \( \omega \in \mathbb{C} \) satisfy \( (1 + \omega \sigma)(1 + \omega \tau) \neq 0 \). Then, the solution of (2.2) with the initial condition \( \zeta(t_0) = \zeta_0 \) is

\[
\zeta(t) = \zeta_0 \frac{e_\omega(t)}{e_\omega(t_0)} + e_\omega(t)S(t),
\]

where \( S \) is given by (2.5).

**Proof.** Lemma 2.1 says that \( e_\omega(t)S(t) \) is a solution to (2.2). By the superposition principle, we see that

\[
\zeta(t) = \zeta_0 \frac{e_\omega(t)}{e_\omega(t_0)} + e_\omega(t)S(t)
\]

is also a solution of (2.2). Moreover, \( \zeta(t_0) = \zeta_0 \) holds because \( S(t_0) = 0 \). Thus, \( \zeta(t) \) is the solution to the initial value problem (2.2) with \( \zeta(t_0) = \zeta_0 \).

3. Technical lemmas

Before giving the main result in the next section, we will discuss some properties of \( \gamma \), as defined in (2.4). Another technical result will also be given for later use.

**Lemma 3.1.** Let \( a, b \in \mathbb{Z} \) with \( a \leq b \). Suppose that \( \omega \in \mathbb{C} \) satisfies \( |(1 + \omega \sigma)(1 + \omega \tau)| \neq 0, 1 \). Then,

\[
\sum_{k=a}^{b-1} |\gamma(k)| = \frac{|\gamma(a-1)| + |\gamma(a-2)| - (|\gamma(b-1)| + |\gamma(b-2)|)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1},
\]

where \( \gamma \) is given in (2.4).

**Proof.** Since

\[
|(1 + \omega \sigma)(1 + \omega \tau)| \sum_{k=a}^{b-1} |\gamma(k)| = \sum_{k=a}^{b-1} |\gamma(k-2)| = \sum_{k=a}^{b-3} |\gamma(k)|
\]

\[
= |\gamma(a-1)| + |\gamma(a-2)| - (|\gamma(b-1)| + |\gamma(b-2)|) + \sum_{k=a}^{b-1} |\gamma(k)|
\]

holds, we get

$$\left| |(1 + \omega \sigma)(1 + \omega \tau)| - 1 \right| \sum_{k=a}^{b-1} |\gamma(k)| = |\gamma(a - 1)| + |\gamma(a - 2)| - (|\gamma(b - 1)| + |\gamma(b - 2)|).$$

Thus, the statement is proven true. □

Define

$$\kappa(t) := \begin{cases} \tau + \sigma|1 + \omega \tau| : \frac{t}{\sigma + \tau} \in \mathbb{Z}, \\ \sigma + \tau|1 + \omega \sigma| : \frac{t - \sigma}{\sigma + \tau} \in \mathbb{Z}. \end{cases} \quad (3.1)$$

Then, we can establish the following lemma.

**Lemma 3.2.** Let \( \nu, \gamma, \) and \( \kappa \) be given by (2.3), (2.4), and (3.1), respectively. Let \( t_1, t_2 \in T_{\sigma, \tau} \) with \( t_1 \leq t_2 \). Suppose that \( \omega \in \mathbb{C} \) satisfies \( |(1 + \omega \sigma)(1 + \omega \tau)| \neq 0, 1 \). Then,

$$\nu(t_2) - 1 \sum_{k=\nu(t_1)}^{\nu(t_2) - 1} |\gamma(k)| = \frac{\kappa(t_1)}{|\kappa(\nu(t_1))|} - \frac{\kappa(t_2)}{|\kappa(\nu(t_2))|}.$$ 

Moreover, the following hold:

(i) If \( |(1 + \omega \sigma)(1 + \omega \tau)| > 1 \), then

$$\sum_{k=\nu(t_1)}^{\infty} |\gamma(k)| = \frac{\kappa(t_1)}{|\kappa(\nu(t_1))|}.$$ 

(ii) If \( 0 < |(1 + \omega \sigma)(1 + \omega \tau)| < 1 \), then

$$\sum_{k=-\infty}^{\nu(t_2) - 1} |\gamma(k)| = \frac{\kappa(t_2)}{|\kappa(\nu(t_2))|}.$$ 

**Proof.** From \( t_1 \leq t_2 \), we have \( \nu(t_1) \leq \nu(t_2) \), and Lemma 3.2 implies

$$\nu(t_2) - 1 \sum_{k=\nu(t_1)}^{\nu(t_2) - 1} |\gamma(k)| = \frac{\gamma(\nu(t_1) - 1) + |\gamma(\nu(t_1) - 2)| - (|\gamma(\nu(t_2) - 1)| + |\gamma(\nu(t_2) - 2)|)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1}.$$ 

We will prove the fact

$$|\gamma(\nu(t) - 1)| + |\gamma(\nu(t) - 2)| = \frac{\kappa(t)}{\kappa(\nu(t))}$$ \quad (3.2)

for \( t \in T_{\sigma, \tau} \). First, we consider the case \( \frac{t}{\sigma + \tau} \in \mathbb{Z} \). From (2.3), we know

$$\nu(t) - 1 = \frac{2t}{\sigma + \tau} - 1 \quad \text{and} \quad \nu(t) - 2 = \frac{2t}{\sigma + \tau} - 2;$$

thus,
\[
\gamma(\nu(t) - 1) = \frac{1}{2} \left[ \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} + \frac{\sigma}{1 + \omega\sigma} \right] \frac{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}}{[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} \\
+ \left( \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} - \frac{\sigma}{1 + \omega\sigma} \right) \frac{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}}{[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} \\
= \frac{\tau}{[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} = \frac{\tau}{e_\omega(t)}
\]

and

\[
\gamma(\nu(t) - 2) = \frac{1}{2} \left[ \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} + \frac{\sigma}{1 + \omega\sigma} \right] \frac{(1 + \omega\sigma)(1 + \omega\tau)}{[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} \\
- \left( \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} - \frac{\sigma}{1 + \omega\sigma} \right) \frac{(1 + \omega\sigma)(1 + \omega\tau)}{[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} \\
= \frac{\sigma(1 + \omega\tau)}{[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} = \frac{\sigma(1 + \omega\tau)}{e_\omega(t)}
\]

Hence, (3.2) holds when \( \frac{t}{\sigma + \tau} \in \mathbb{Z} \).

Next, we consider the case \( \frac{t - \sigma}{\sigma + \tau} \in \mathbb{Z} \). From (2.3), we see that

\[
\nu(t) - 1 = \frac{2(t - \sigma)}{\sigma + \tau} \quad \text{and} \quad \nu(t) - 2 = \frac{2(t - \sigma)}{\sigma + \tau} - 1.
\]

Therefore,

\[
\gamma(\nu(t) - 1) = \frac{1}{2} \left[ \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} + \frac{\sigma}{1 + \omega\sigma} \right] \left( \frac{1}{(1 + \omega\sigma)(1 + \omega\tau)} \right) \frac{t - \sigma}{\sigma + \tau} \\
- \left( \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} - \frac{\sigma}{1 + \omega\sigma} \right) \left( \frac{1}{(1 + \omega\sigma)(1 + \omega\tau)} \right) \frac{t - \sigma}{\sigma + \tau} \\
= \frac{\sigma}{(1 + \omega\sigma)[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} = \frac{\sigma}{e_\omega(t)}
\]

and

\[
\gamma(\nu(t) - 2) = \frac{1}{2} \left[ \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} + \frac{\sigma}{1 + \omega\sigma} \right] \sqrt{(1 + \omega\sigma)(1 + \omega\tau)} \left( \frac{1}{[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} \right) \\
+ \left( \frac{\tau}{\sqrt{(1 + \omega\sigma)(1 + \omega\tau)}} - \frac{\sigma}{1 + \omega\sigma} \right) \sqrt{(1 + \omega\sigma)(1 + \omega\tau)} \left( \frac{1}{[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} \right) \\
= \frac{\tau}{[(1 + \omega\sigma)(1 + \omega\tau)]^{\frac{1}{\pi^+}}} = \frac{\tau(1 + \omega\sigma)}{e_\omega(t)}.
\]

Hence, (3.2) holds for all \( t \in T_{\sigma, \tau} \).

Next, we consider case (i). If \(|(1 + \omega\sigma)(1 + \omega\tau)| > 1\), then

\[
\lim_{t \to \infty} |e_\omega(t)| = \infty.
\]

This, together with the above mentioned fact, implies that
\[ \sum_{k=\nu(t_1)}^{\infty} |\gamma(k)| = \frac{s(t_1)}{|e_{\nu}(t_1)|} \frac{1}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1}. \]

Using the same argument we have assertion (ii). This completes the proof. \( \square \)

This final lemma will provide a notation for the best HUS constant, to be found in the subsequent section.

**Lemma 3.3.** Let \( \omega \in \mathbb{C} \) satisfy \( |(1 + \omega \sigma)(1 + \omega \tau)| \neq 0, 1 \). Then,

\[
\max\left\{ \sigma + \tau |1 + \omega \sigma|, \tau + \sigma |1 + \omega \tau| \right\} \frac{1}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} = K_{\sigma\tau},
\]

where

\[
K_{\sigma\tau} := \begin{cases} 
\frac{\sigma + \tau |1 + \omega \sigma|}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} & \text{if } \sigma \geq \tau, \\
\frac{\tau + \sigma |1 + \omega \tau|}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} & \text{if } \tau > \sigma.
\end{cases}
\]

(3.3)

**Proof.** Define the function

\[
g(x) := x + \left| \frac{1}{\sigma} + \omega \right| - \left| x + \frac{1}{\sigma} + \omega \right|
\]

for \( x \in \mathbb{R} \). From

\[
g(x) = x + \left| \frac{1}{\sigma} + \omega \right| - \sqrt{\left( x + \frac{1}{\sigma} + \text{Re}(\omega) \right)^2 + (\text{Im}(\omega))^2},
\]

we have

\[
g'(x) = \frac{\sqrt{(x + \frac{1}{\sigma} + \text{Re}(\omega))^2 + (\text{Im}(\omega))^2} - \sqrt{(x + \frac{1}{\sigma} + \text{Re}(\omega))^2 + (\text{Im}(\omega))^2}}{\sqrt{(x + \frac{1}{\sigma} + \text{Re}(\omega))^2 + (\text{Im}(\omega))^2}} \geq 0
\]

for \( x \in \mathbb{R} \). This, together with \( g(0) = 0 \), implies that \( g(x) \geq 0 \) if \( x \geq 0 \), and \( g(x) \leq 0 \) if \( x < 0 \). We notice that

\[
\sigma \tau g\left( \frac{\sigma - \tau}{\sigma \tau} \right) = \sigma \tau g\left( \frac{1}{\tau} - \frac{1}{\sigma} \right) = \sigma + \tau |1 + \omega \sigma| - (\tau + \sigma |1 + \omega \tau|),
\]

and thus, we conclude that

\[
\sigma + \tau |1 + \omega \sigma| - (\tau + \sigma |1 + \omega \tau|) \geq 0
\]

if \( \sigma - \tau \geq 0 \), and

\[
\sigma + \tau |1 + \omega \sigma| - (\tau + \sigma |1 + \omega \tau|) \leq 0
\]

if \( \sigma - \tau < 0 \). This ends the proof. \( \square \)
4. Main results

The main theorem in this paper is as follows; it draws from the preliminary results given earlier.

**Theorem 4.1.** Let \( t_0 \in \mathbb{T}_{\sigma,\tau}, \ z_0 \in \mathbb{C}, \) and let \( \omega \in \mathbb{C} \) satisfy \(|(1 + \omega \sigma)(1 + \omega \tau)| \neq 0, 1. \) Let \( \kappa \) be given by (3.1) and \( K_{\sigma \tau} \) by (3.3). Suppose that \( \zeta : \mathbb{T} \to \mathbb{C} \) satisfies

\[
|\zeta'(t) - \omega \zeta(t)| \leq \varepsilon
\]

for \( t \in \mathbb{T}^1, \) where \( \varepsilon > 0 \) is a fixed arbitrary constant. Then, one of the following holds:

(i) If \(|(1 + \omega \sigma)(1 + \omega \tau)| > 1 \) and \( t^* := \max \mathbb{T} \) exists, then any solution \( z \) of (1.1) with

\[
|\zeta(t^*) - z(t^*)| < \frac{\varepsilon K(t^*)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1}
\]

satisfies

\[
|\zeta(t) - z(t)| < \varepsilon K_{\sigma \tau}, \quad t \in \mathbb{T}.
\]

(ii) If \(|(1 + \omega \sigma)(1 + \omega \tau)| > 1 \) and \( \max \mathbb{T} \) does not exist, then \( \lim_{t \to \infty} \frac{\zeta(t)}{e_{\omega}(t)} \) exists, and the function \( z \) given by

\[
z(t) := \left( \lim_{t \to \infty} \frac{\zeta(t)}{e_{\omega}(t)} \right) e_{\omega}(t)
\]

is the unique solution of (1.1) such that

\[
|\zeta(t) - z(t)| \leq \varepsilon K_{\sigma \tau}, \quad t \in \mathbb{T}.
\]

(iii) If \( 0 < |(1 + \omega \sigma)(1 + \omega \tau)| < 1 \) and \( t_* := \min \mathbb{T} \) exists, then any solution \( z \) of (1.1) with

\[
|\zeta(t_*) - z(t_*)| < \frac{\varepsilon K(t_*)}{1 - |(1 + \omega \sigma)(1 + \omega \tau)|}
\]

satisfies

\[
|\zeta(t) - z(t)| < \varepsilon |K_{\sigma \tau}|, \quad t \in \mathbb{T}.
\]

(iv) If \( 0 < |(1 + \omega \sigma)(1 + \omega \tau)| < 1 \) and \( \min \mathbb{T} \) does not exist, then \( \lim_{t \to -\infty} \frac{\zeta(t)}{e_{\omega}(t)} \) exists, and

\[
z(t) := \left( \lim_{t \to -\infty} \frac{\zeta(t)}{e_{\omega}(t)} \right) e_{\omega}(t)
\]

is the unique solution of (1.1) such that

\[
|\zeta(t) - z(t)| \leq \varepsilon |K_{\sigma \tau}|, \quad t \in \mathbb{T}.
\]
Proof. Let \( \varepsilon > 0 \) be given. Suppose that \( \zeta : \mathbb{T} \to \mathbb{C} \) satisfies

\[
|\zeta^\Delta(t) - \omega \zeta(t)| \leq \varepsilon
\]

for \( t \in \mathbb{T}^\uparrow \). Define

\[
f(t) := \zeta^\Delta(t) - \omega \zeta(t), \quad |f(t)| \leq \varepsilon. \tag{4.1}
\]

Let \( t_0 \in \mathbb{T} \) and \( \zeta_0 \in \mathbb{C} \). Then, we can solve this equation. By Lemma 2.2, the solution of the initial value problem (2.2) with \( \zeta(t_0) = \zeta_0 \) is expressed as follows:

\[
\zeta(t) = \zeta_0 \frac{e_\omega(t)}{e_\omega(t_0)} + e_\omega(t)S(t),
\]

where \( S(t) \) is given in (2.5).

First, we consider case (i). Let \( t^* = \max \mathbb{T} \). Then, we can find \( \zeta \) as

\[
\zeta(t) = \zeta(t^*) \frac{e_\omega(t)}{e_\omega(t^*)} + e_\omega(t)S(t)
\]

for \( t \leq t^* \) and \( t \in \mathbb{T}_{\sigma, \tau} \). Now we consider the function \( z : \mathbb{T} \to \mathbb{C} \) satisfying (1.1) and the inequality

\[
|\zeta(t^*) - z(t^*)| < \frac{\varepsilon \kappa(t^*)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1}. \tag{4.2}
\]

Since \( z \) is a solution to (1.1), we can rewrite \( z \) as

\[
z(t) = z(t^*) \frac{e_\omega(t)}{e_\omega(t^*)}
\]

for all \( t \leq t^* \). Therefore,

\[
\zeta(t) - z(t) = \frac{e_\omega(t)}{e_\omega(t^*)} \left( \zeta(t^*) - z(t^*) + e_\omega(t^*)S(t) \right)
\]

\[
= \frac{e_\omega(t)}{e_\omega(t^*)} \left( \zeta(t^*) - z(t^*) + e_\omega(t^*) \sum_{k=\nu(t)}^{\nu(t^*)-1} |\gamma(k)| |f(\nu^{-1}(k))| \right)
\]

for \( t \leq t^* \). From (4.1) and (4.2), we have

\[
|\zeta(t) - z(t)| \leq \left| \frac{e_\omega(t)}{e_\omega(t^*)} \right| \left( |\zeta(t^*) - z(t^*)| + |e_\omega(t^*)| \sum_{k=\nu(t)}^{\nu(t^*)-1} |\gamma(k)| |f(\nu^{-1}(k))| \right)
\]

\[
< \varepsilon \left| \frac{e_\omega(t)}{e_\omega(t^*)} \right| \left( \frac{\kappa(t^*)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} + |e_\omega(t^*)| \sum_{k=\nu(t)}^{\nu(t^*)-1} |\gamma(k)| \right).
\]

In addition, using Lemma 3.2, we see that

\[ |\zeta(t) - z(t)| \leq \varepsilon \left| \frac{e_\omega(t)}{e_\omega(t^*)} \right| \left( \frac{\kappa(t^*)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} + \left| \frac{e_\omega(t^*)}{e_\omega(t^*)} \right| \right) \]

\[ = \frac{\varepsilon \kappa(t)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} \]

\[ \leq \frac{\varepsilon \max\{\sigma + \tau|1 + \omega \sigma|, \tau + \sigma|1 + \omega \tau|\}}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} = \varepsilon K_{\sigma \tau} \]

for all \( t \leq t^* \). Thus, assertion (i) is true, by Lemma 3.3 and (3.3).

Next, we consider case (ii). Suppose that \( \max T \) does not exist. Let \( t_0 \in T \). Then, \( \zeta \) can be written as

\[ \zeta(t) = \zeta(t_0) \frac{e_\omega(t)}{e_\omega(t_0)} + e_\omega(t)S(t) \]

for all \( t \in T \). Using Lemma 3.2 and the assumptions \(|(1 + \omega \sigma)(1 + \omega \tau)| > 1\) and \( |f(t)| \leq \varepsilon \), we see that

\[ \left| \sum_{k=\nu(t_0)}^{\nu(t)-1} \gamma(k) f(\nu^{-1}(k)) \right| \leq \varepsilon \sum_{k=\nu(t_0)}^{\nu(t)-1} |\gamma(k)| = \frac{\varepsilon \left( \frac{\kappa(t_0)}{|e_\omega(t_0)|} - \frac{\kappa(t)}{|e_\omega(t)|} \right)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} \]

for \( t \geq t_0 \). This says that \( \lim_{t \to \infty} S(t) \) exists. Let

\[ z_0 := \frac{\zeta(t_0)}{e_\omega(t_0)} + \lim_{t \to \infty} S(t). \]

This implies that \( \lim_{t \to \infty} \frac{\zeta(t)}{e_\omega(t)} \) exists. Now we consider the function

\[ z(t) := z_0 e_\omega(t) \]

for \( t \in T \). Then, \( z : T \to \mathbb{C} \) is a solution to (1.1), and satisfies

\[ \zeta(t) - z(t) = e_\omega(t) \left( \frac{\zeta(t_0)}{e_\omega(t_0)} - z_0 + S(t) \right) = e_\omega(t) \left( S(t) - \lim_{t \to \infty} S(t) \right) \]

\[ = e_\omega(t) \left( - \sum_{k=\nu(t)}^{\infty} \gamma(k) f(\nu^{-1}(k)) \right) \]

for all \( t \in T \). By using Lemma 3.2, we obtain

\[ |\zeta(t) - z(t)| \leq |e_\omega(t)| \left| \sum_{k=\nu(t)}^{\infty} \gamma(k) f(\nu^{-1}(k)) \right| \leq |e_\omega(t)| \sum_{k=\nu(t)}^{\infty} |\gamma(k) f(\nu^{-1}(k))| \]

\[ \leq \varepsilon |e_\omega(t)| \sum_{k=\nu(t)}^{\infty} |\gamma(k)| = \frac{\varepsilon K(t)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} \]

\[ \leq \frac{\varepsilon \max\{\sigma + \tau|1 + \omega \sigma|, \tau + \sigma|1 + \omega \tau|\}}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} = \varepsilon K_{\sigma \tau} \]

for all \( t \in T \). The first part of the result follows from Lemma 3.3 and (3.3).
Next, we will discuss the uniqueness of the solution. Assume that there exists a solution $z_1 : \mathbb{T} \to \mathbb{C}$ of (1.1) that is different from the above one, and satisfies the inequality

$$|\zeta(t) - z_1(t)| \leq \varepsilon K_{\sigma \tau}$$

for all $t \in \mathbb{T}$. That is, $z_1$ is rewritten by $z_1(t) := z_1 e_{\omega}(t)$, where $z_1 \neq z_0$. Using the above inequality, we have

$$|z_0 - z_1| e_{\omega}(t) = |z(t) - z_1(t)| \leq |z(t) - \zeta(t)| + |\zeta(t) - z_1(t)| \leq 2\varepsilon K_{\sigma \tau}$$

for $t \in \mathbb{T}$. Note here that the assumption $|(1 + \omega \sigma)(1 + \omega \tau)| > 1$ implies $\lim_{t \to \infty} |z_0 - z_1| e_{\omega}(t) = \infty$, and so that, this is a contradiction. Thus, assertion (ii) is true.

The proofs of (iii) and (iv) are omitted because they are shown using the same method as (i) and (ii), respectively. This completes the proof. □

By Theorem 4.1, we get the following result immediately concerning the HUS constant, including definitive statements about the HUS constant being the best (minimal) constant.

**Theorem 4.2.** Let $\omega \in \mathbb{C}$ satisfy $|(1 + \omega \sigma)(1 + \omega \tau)| \neq 0, 1$. Then, (1.1) has HUS with an HUS constant

$$K = |K_{\sigma \tau}|$$

on $\mathbb{T}$, for $K_{\sigma \tau}$ in (3.3). Moreover,

(i) if $|(1 + \omega \sigma)(1 + \omega \tau)| > 1$ and $\max \mathbb{T}$ does not exist, then

$$K = K_{\sigma \tau}$$

is the best HUS constant;

(ii) if $0 <|(1 + \omega \sigma)(1 + \omega \tau)| < 1$ and $\min \mathbb{T}$ does not exist, then

$$K = -K_{\sigma \tau}$$

is the best HUS constant.

**Proof.** The HUS constant $K$ follows directly from Theorem 4.1, using Lemma 3.3 and (3.3).

Let $\omega \in \mathbb{C}$ satisfy $|(1 + \omega \sigma)(1 + \omega \tau)| \neq 0, 1$. Let $\kappa$ be given by (3.1), and let $\varepsilon > 0$ be a fixed arbitrary constant. Take $f : \mathbb{T} \to \mathbb{C}$ to be given by

$$f(t) = \frac{\varepsilon |\gamma(\nu(t))|}{\gamma(\nu(t))} \quad \text{or} \quad f^{-1}(k) = \frac{\varepsilon |\gamma(k)|}{\gamma(k)},$$

where $\nu$ is given in (2.3) and $\gamma$ is given in (2.4). Then, for $S$ given in (2.5), we have

$$S(t) = \begin{cases} \varepsilon \sum_{k=\nu(t_0)}^{\nu(t)-1} |\gamma(k)| & : t \geq t_0, t \in \mathbb{T}_{\sigma, \tau}, \\
-\varepsilon \sum_{k=\nu(t)}^{\nu(t_0)-1} |\gamma(k)| & : t \leq t_0, t \in \mathbb{T}_{\sigma, \tau}, \\
\end{cases}$$

for $t \in \mathbb{T}$. 

(i) Suppose \(|1 + \omega \sigma)(1 + \omega \tau)| > 1\) and \(\max T\) does not exist. Let \(t_0 \in T\), and let \(\zeta : T \to \mathbb{C}\) be given by

\[
\zeta(t) = \zeta(t_0) \frac{e_\omega(t)}{e_\omega(t_0)} + e_\omega(t)S(t);
\]

clearly

\[
|\zeta^\Delta(t) - \omega \zeta(t)| = \varepsilon
\]

for all \(t \in T\). We now proceed as in the proof of the above result, Theorem 4.1 (ii). Let

\[
z(t) := z_0 e_\omega(t), \quad z_0 := \frac{\zeta(t_0)}{e_\omega(t_0)} + \lim_{t \to \infty} S(t),
\]

for \(t \in T\). Then, \(z : T \to \mathbb{C}\) is a solution to (1.1), and together with \(\zeta\) satisfies

\[
|\zeta(t) - z(t)| = \varepsilon |e_\omega(t)| \sum_{k=\omega(t)}^\infty |\gamma(k)| = \frac{\varepsilon \kappa(t)}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1}
\]

\[
= \varepsilon \left\{ \frac{\tau + \sigma |1 + \omega \tau|}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} : \frac{\tau}{\sigma + \tau} \in \mathbb{Z}, \right. \\
\left. \frac{\sigma + \tau |1 + \omega \sigma|}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} : \frac{\sigma - \tau}{\sigma + \tau} \in \mathbb{Z} \right\}
\]

for all \(t \in T\). Consequently, the minimal HUS constant is at least \(K_{\sigma \tau}\). Since \(K = |K_{\sigma \tau}|\), the result follows.

Case (ii), based on the proof of Theorem 4.1 (iv), would be similar and is omitted. 

\[\square\]

**Remark 4.3.** If \(\omega\) is real with \(\omega > 0\), then

\[
K = \frac{\max \{\sigma + \tau |1 + \omega \sigma|, \sigma + \tau |1 + \omega \tau|\}}{|1 - |(1 + \omega \sigma)(1 + \omega \tau)||} = \frac{\sigma + \tau + \omega \sigma \tau}{\omega \tau + \omega \sigma + \omega^2 \sigma \tau} = \frac{1}{\omega}.
\]

If \(\omega\) is real with \(\max \{-\frac{1}{\sigma}, -\frac{1}{\tau}\} < \omega < 0\) such that \(T\) is unbounded below, then

\[
K = \frac{\max \{\sigma + \tau |1 + \omega \sigma|, \sigma + \tau |1 + \omega \tau|\}}{|1 - |(1 + \omega \sigma)(1 + \omega \tau)||} = \frac{\sigma + \tau + \omega \sigma \tau}{|\omega \tau + \omega \sigma + \omega^2 \sigma \tau|} = \frac{1}{|\omega|}.
\]

These constants are the best (minimal) HUS constants for the real version of (1.1) on \(T\); see [2].

For a second example, suppose \(\sigma > (3 + 2\sqrt{2}) \tau\) and \(\max T\) does not exist. Then, \(\sigma^2 + \tau^2 - 6 \sigma \tau > 0\); set

\[
\omega_+ = -\frac{\sigma - \tau + \sqrt{\sigma^2 + \tau^2 - 6 \sigma \tau}}{2 \sigma \tau}, \quad \omega_- = -\frac{\sigma - \tau - \sqrt{\sigma^2 + \tau^2 - 6 \sigma \tau}}{2 \sigma \tau}.
\]

If \(\omega \in \mathbb{R}\) such that \(\omega_- < \omega < \omega_+\), then \(|(1 + \omega \sigma)(1 + \omega \tau)| > 1\) \((1 + \omega \sigma)(1 + \omega \tau) < -1\) with \(1 + \omega \sigma < 0\) and \(1 + \omega \tau > 0\),

\[
\sigma + \tau |1 + \omega \sigma| > \tau + \sigma |1 + \omega \tau|,
\]

and

\[
K_{\sigma \tau} = \frac{\sigma + \tau |1 + \omega \sigma|}{|(1 + \omega \sigma)(1 + \omega \tau)| - 1} = \frac{\sigma - \tau (1 + \omega \sigma)}{|2 + \omega (\sigma + \tau) + \omega^2 \sigma \tau|},
\]

which agrees with the best HUS constant for (1.1) in the case of \(\omega\) real given in [3, Theorem 1.3 B].
For a third example, if $\sigma = \tau = h$, then
\[
K = \frac{\max\{\sigma + \tau|1 + \omega\sigma|, \tau + \sigma|1 + \omega\tau|\}}{|1 - |(1 + \omega\sigma)(1 + \omega\tau)||} = \frac{h + h|1 + \omega h|}{|1 - |1 + \omega h|^2|} = \frac{h}{|1 - |1 + \omega h|^2|}.
\]
This constant is the best (minimal) HUS constant for $h$-difference equation (1.3) on $h\mathbb{Z}$; see [4].

Finally, let $\omega = 1 + i$, $\sigma = 2$, $\tau = 1$. Then, $|(1 + \omega\sigma)(1 + \omega\tau)| > 1$, and
\[
\sqrt{13} + 2 = \sigma + \tau|1 + \omega\sigma| > \tau + \sigma|1 + \omega\tau| = 2\sqrt{5} + 1,
\]
so that, if $T$ is unbounded above, $K_{\sigma\tau} = \frac{\sqrt{13}+2}{\sqrt{65}-1}$ is the best HUS constant for (1.1).

**Theorem 4.4.** Let $\omega \in \mathbb{C}$ satisfy $|(1 + \omega\sigma)(1 + \omega\tau)| = 1$. If max $T$ or min $T$ does not exist, then (1.1) does not have HUS on $T$.

**Proof.** As $\omega \in \mathbb{C}$ satisfies $|(1 + \omega\sigma)(1 + \omega\tau)| = 1$, the exponential function $e_\omega$ given by (2.1) clearly satisfies
\[
0 < \min\{1, |1 + \omega\sigma|\} \leq |e_\omega(t)| \leq \max\{1, |1 + \omega\sigma|\}
\]
for all $t \in T$. In the case that $\omega \in \mathbb{R}$ and max $T$ does not exist, by [2, Theorem 3.10 (ii)], we see that (1.1) is not Hyers–Ulam stable. Now, let us directly show that it does not have HUS in all cases $\omega \in \mathbb{C}$ satisfying $|(1 + \omega\sigma)(1 + \omega\tau)| = 1$. Let $M := \max\{1, |1 + \omega\sigma|\}$ and
\[
\zeta(t) := \frac{\varepsilon}{M}t e_\omega(t)
\]
for all $t \in T$. Then, we have
\[
\zeta^\Delta(t) = \omega \zeta(t) + \frac{\varepsilon}{M}e_\omega(t) \begin{cases} (1 + \omega\sigma) & : \frac{t - \sigma}{\sigma + \tau} \in \mathbb{Z}, \\ (1 + \omega\tau) & : \frac{t - \tau}{\sigma + \tau} \in \mathbb{Z}. \end{cases}
\]
This, together with $|(1 + \omega\sigma)(1 + \omega\tau)| = 1$, implies that
\[
|\zeta^\Delta(t) - \omega \zeta(t)| = \frac{\varepsilon}{M} \begin{cases} |1 + \omega\sigma| & : \frac{t - \sigma}{\sigma + \tau} \in \mathbb{Z}, \\ 1 & : \frac{t - \tau}{\sigma + \tau} \in \mathbb{Z}. \end{cases} \leq \varepsilon
\]
for all $t \in T$.

Now we consider the general solution to (1.1) which given by $z(t) = ce_\omega(t)$, where $c \in \mathbb{C}$ is an arbitrary constant. Then,
\[
|\zeta(t) - z(t)| = \left|\frac{\varepsilon}{M}t - c\right| |e_\omega(t)| \geq \left|\frac{\varepsilon}{M}t - c\right| \min\{1, |1 + \omega\sigma|\}
\]
for all $t \in T$. If max $T$ does not exist, then we have $\lim_{t \to \infty} |\zeta(t) - z(t)| = \infty$, thus, (1.1) is not HUS on $T$. Similarly, in the case where min $T$ does not exist, it is not HUS when $t \to -\infty$. This completes the proof. $\Box$

If $T = T_{\sigma, \tau}$, then the following necessary and sufficient condition is obtained by Theorems 4.2 and 4.4.

**Theorem 4.5.** Let $T = T_{\sigma, \tau}$, and let $\omega \in \mathbb{C}$ satisfy $|(1 + \omega\sigma)(1 + \omega\tau)| \neq 0$. Then, (1.1) has HUS on $T$ if and only if $|(1 + \omega\sigma)(1 + \omega\tau)| \neq 1$. 
5. Visualizing stability and instability in the complex plane

We provide a brief visualization of what is happening in the complex plane in relation to the earlier stability results, in Fig. 1 graphs (i)–(iv). In (2.1), reparametrize according to the polar formula $\rho e^{i\varphi} = (1 + \omega\sigma)(1 + \omega\tau)$ and solve for $\omega \in \mathbb{C}$. Note here that

$$(\sigma - \tau)^2 - 4\sigma\tau = \sigma^2 + \tau^2 - 6\sigma\tau,$$

which is the discriminant of $(1 + \omega\sigma)(1 + \omega\tau) = -1$ found in Remark 4.3, and $\tau = (3 - 2\sqrt{2})\sigma$ solves $(\sigma - \tau)^2 - 4\sigma\tau = 0$. If $\rho = 1$, then (1.1) is unstable, whereas (1.1) is stable for $\rho \in (0, 1) \cup (1, \infty)$. In the following graphs, blue ($\rho > 1$) and green ($\rho < 1$) are HUS stable values of $\omega$, while red ($\rho = 1$) is unstable. For each graph, $\varphi \in [-\pi, \pi]$.

Fig. 1. (i) $0 < \rho < \frac{(\sigma - \tau)^2}{4\sigma\tau} < 1$, $\rho^{-1} > 1$, $\sigma = 1$, $\tau = 0.2$. (ii) $\rho = \frac{3}{2}$, $\rho^{-1} = \frac{2}{3}$, $\sigma = 1$, and $\tau = (3 - 2\sqrt{2})\sigma$. (iii) $\rho = \frac{(\sigma - \tau)^2}{4\sigma\tau} > 1$, $0 < \rho^{-1} < 1$, $\sigma = 1$, $\tau = 0.14$. (iv) $\rho > \frac{(\sigma - \tau)^2}{4\sigma\tau} > 1$, $0 < \rho^{-1} < 1$, $\tau = 1$, $\sigma > \tau(3 + 2\sqrt{2})$. (For interpretation of the colors in the figure, the reader is referred to the web version of this article.)
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