Results in Fixed Point Theory and Applications

Layered Monotonic Fixed Point Theorem

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Abstract: In this paper, we decompose an operator as the sum of an increasing operator and a decreasing operator, with a condition that the change in the increasing component is greater than the change in the decreasing component on a given set, which allows us to use a monotonic iterative method to find a fixed point for the operator. We illustrate the method by applying it to the second-order right focal boundary value problem, proving the existence of a positive solution.

Keywords: fixed-point theorems, monotonic, alternate inversion, iterative.

MSC: 47H10.

1 Introduction

Once one knows that an operator \(T\) has a fixed point in a given set, the question becomes how to find that fixed point. The simplest method to find a fixed point for an operator is to apply a monotonic iterative method when the conditions are met to apply monotone iterative methods. See Zeidler \([8, p283]\) for a thorough treatment of monotone iterative methods which are credited to Krasnoselskii \([4]\) with refinements and extensions by many others. An application of the monotone iterative method is simple once one has a bounded, invariant set containing a sub-solution or a super-solution. In this paper we show how one

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can use the ideas presented in the Layered Compression-Expansion Fixed Point Theorem [1] to find a set in which to create a monotonic sequence by iteration, which will converge to a fixed point. The key to our argument is to write the operator $T$ as a sum of an increasing operator $R$ and a decreasing operator $S$, so

$$T = R + S,$$

with

$$R : Y + Z \rightarrow Y \quad \text{and} \quad S : Y + Z \rightarrow Z$$

such that we can create a monotonic sequence,

$$x_{n+1} = Tx_n \quad \text{for} \quad n \geq 0 \quad \text{with} \quad x_0 \in Y + Z,$$

by iteration. We conclude with an application to the boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0,1),$$

$$x(0) = 0 = x'(1),$$

where $f : \mathbb{R} \rightarrow [0,\infty)$ is differentiable and

$$f = f_+ + f_-$$

with $f_+ : \mathbb{R} \rightarrow [0,\infty)$ being an increasing, differentiable function and $f_- : \mathbb{R} \rightarrow [0,\infty)$ being a decreasing, differentiable function. The solutions of this boundary value problem are the fixed points of an operator

$$Tx(t) := \int_0^1 G(t, \tau) f(x(\tau)) d\tau,$$

which we will write as the sum of the increasing operator $R$ and the decreasing operator $S$ given by

$$(R(r+s))(t) := \int_0^1 G(t, \tau) f_+(r(\tau) + s(\tau)) d\tau$$

and

$$(S(r+s))(t) := \int_0^1 G(t, \tau) f_-(r(\tau) + s(\tau)) d\tau$$

with $x(t) = r(t) + s(t)$.

While there is a lot of work in the literature concerning monotone iterative techniques to yield bounds on solutions including a very thorough summary by Ladde, Lakshmikantham and Vatsala [5] and recent applications to causal operators and fractional problems [2, 6, 7], there are no papers concentrated on finding intervals to apply the monotone iterative method to find a fixed point and not just approximations or bounds of fixed points. Utilizing the Layered Compression-Expansion Fixed Point Theorem [1] we outline a method for finding closed intervals that can be used to apply the monotone iterative method to find a fixed point by iteration to an operator. Our results rely on finding an invariant set where the change in the increasing component is greater than the change in the decreasing component that also contains a sub-solution.
2 Main Results

The backbone of the Layered Compression-Expansion Fixed Point Theorem [1] is Theorem 2.1 below, which provides a nonstandard method of finding a fixed point for the sum of two operators. Our main result here, Theorem 2.2, which is proven in the same manner as the classical monotonic iterative theorems, applies the techniques of Theorem 2.1 to verify the existence of a fixed point which we can find by iteration. Our results rely on a working knowledge of the definitions for concave and convex functionals, cones, and monotonic norm (normal cones) which can be found in [3, 8].

**Theorem 2.1.** Suppose $P$ is a cone in a real Banach space $E$, $R : P \to P$ and $S : P \to P$ are operators, and $T = R + S$. If

$$M : P \times P \to P \times P$$

defined by

$$M(r, s) = (R(r + s), S(r + s))$$

has a fixed point $(r^*, s^*)$, then $x^* = r^* + s^*$ is a fixed point of $T$.

Our main result relies on decomposing the operator $T$ into the sum of an increasing operator $R$ and a decreasing operator $S$, with a condition that can be interpreted as the change in the increasing component is greater than the change in the decreasing component of the operator. This results in a monotonic iterative method to find a fixed point for the operator $T$.

**Theorem 2.2.** Suppose $Y$ and $Z$ are bounded, closed subsets of the cone $P$ in the Banach space $E$ with monotonic norm, and

$$R : Y + Z \to Y$$

is an increasing, completely continuous operator,

$$S : Y + Z \to Z$$

is a decreasing, completely continuous operator, and $T = R + S$. Further, suppose that there are elements $r_0 \in Y$ and $s_0 \in Z$, and for all $n \geq 1$

$$r_n = R(r_{n-1} + s_{n-1}) \quad \text{and} \quad s_n = S(r_{n-1} + s_{n-1}),$$

with

$$r_n - r_{n-1} \geq s_{n-1} - s_n.$$

Then there exists a fixed point $x^* = r^* + s^*$ for $T$ with $(r^*, s^*) \in Y \times Z$. Moreover, the sequence $\{r_n\}$ increases to $r^*$ and the sequence $\{s_n\}$ decreases to $s^*$.

**Proof.** Let $r_0 \in Y$ and $s_0 \in Z$; define the sequences $\{r_n\}_{n=1}^\infty$ and $\{s_n\}_{n=1}^\infty$ by

$$r_n = R(r_{n-1} + s_{n-1}) \quad \text{and} \quad s_n = S(r_{n-1} + s_{n-1}),$$

respectively, and suppose that

$$r_n - r_{n-1} \geq s_{n-1} - s_n.$$
for \( n \geq 1 \). Thus
\[
 r_n + s_n \geq r_{n-1} + s_{n-1},
\]
and hence
\[
 r_{n+1} = R(r_n + s_n) \geq R(r_{n-1} + s_{n-1}) = r_n
\]
for all \( n \geq 1 \), since \( R \) is increasing on \( Y + Z \) and
\[
 s_{n+1} = S(r_n + s_n) \leq S(r_{n-1} + s_{n-1}) = s_n
\]
for all \( n \geq 1 \), since \( S \) is decreasing on \( Y + Z \). Therefore we have that
\[
\{r_n\}_{n=1}^\infty \quad \text{and} \quad \{s_n\}_{n=1}^\infty
\]
are monotonic sequences in the bounded, closed subset \( Y + Z \) of the cone \( P \). Thus, since \( R \) and \( S \) are completely continuous with
\[
 r_{n+1} = R(r_n + s_n) \quad \text{and} \quad s_{n+1} = S(r_n + s_n),
\]
there are subsequences \( \{s_{n_k}\}_{k=1}^\infty \) and \( \{r_{n_k}\}_{k=1}^\infty \) with
\[
 r_{n_k} \to r^* \in Y \quad \text{and} \quad s_{n_k} \to s^* \in Z.
\]
The norm is monotonic, thus we actually have
\[
 r_n \to r^* \in Y \quad \text{and} \quad s_n \to s^* \in Z,
\]
since for any \( k \in \mathbb{N} \), when \( j \geq n_k \) we have
\[
 ||r^* - r_j|| \leq ||r^* - r_{n_k}|| \quad \text{because} \quad r^* - r_j \leq r^* - r_{n_k}
\]
and
\[
 ||s_j - s^*|| \leq ||s_{n_k} - s^*|| \quad \text{because} \quad s_j - s^* \leq s_{n_k} - s^*.
\]
Therefore
\[
 r^* = R(r^* + s^*) \quad \text{and} \quad s^* = S(r^* + s^*),
\]
and hence by Theorem 2.1 we have that \( x^* = r^* + s^* \) is a fixed point for the operator \( T \).

Note that the Layered Compression-Expansion Fixed Point Theorem provides the context to say that the increasing component of \( x_n - x_{n-1} \), namely
\[
 r_n = r_{n-1},
\]
is greater than the decreasing component of \( x_n - x_{n-1} \), namely
\[
 s_{n-1} - s_n,
\]
which is what it takes to say that
\[
 x_n \geq x_{n-1},
\]
and which enables us to apply the iterative argument of Theorem 2.2. That is, the sequence
\[
 x_{n+1} = Tx_n, \quad n \geq 0,
\]
is an increasing sequence which converges to a fixed point of \( T \).
3 Application

In this section, we will verify the existence of a positive solution by applying Theorem 2.2 to the right focal boundary value problem

\[ x''(t) + f(x(t)) = 0, \quad t \in (0, 1), \]  
\[ x(0) = 0 = x'(1), \]  

where \( f : \mathbb{R} \to [0, \infty) \) is differentiable and

\[ f = f_+ + f_- \]

with \( f_+ : \mathbb{R} \to [0, \infty) \) being an increasing, differentiable function and \( f_- : \mathbb{R} \to [0, \infty) \) being a decreasing, differentiable function. If \( x \) is a fixed point of the operator

\[ Tx(t) := \int_0^1 G(t, \tau) f(x(\tau)) d\tau, \]

then \( x \) is a solution of the boundary value problem (1), (2).

Let \( E = C[0, 1] \) and \( P \) be the subset of nonnegative, nondecreasing, concave elements of \( C[0, 1] \). Then \( P \) is a cone in the Banach space \( E \) accompanied by the sup-norm (which is a monotonic norm on \( P \)). For \( x \in P \), define the convex functional \( \beta \) and the concave functional \( \alpha \) on \( P \) by

\[ \beta(x) := \max_{t \in [0,1]} x(t) = x(1) \quad \text{and} \quad \alpha(x) := \min_{t \in [\frac{1}{4}, 1]} x(t) = x \left( \frac{1}{4} \right). \]

We are now ready to prove the existence of a positive solution to (1), (2) by applying Theorem 2.2.

**Theorem 3.1.** Assume \( a, b, c \) and \( d \) are real numbers with \( b > a, d > c \). Suppose that \( f_- : [0, \infty) \to [0, \infty) \) is a decreasing, differentiable function and \( f_+ : [0, \infty) \to [0, \infty) \) is an increasing, differentiable function. Also, suppose that

(a) \( \frac{16c}{3} \leq f_-(b + d) \) and \( f_-(0) \leq 2d \),

(b) \( \frac{16a}{3} \leq f_+(a + c) \) and \( f_+(b + d) \leq 2b \),

(c) \( \min_{z \in [a + c, b + d]} f_+^1(z) \geq \max_{w \in [b, d]} |f_-^1(w)| \), and

(d) \( \min_{z \in [a + c, b + d]} f_+^1(z) \geq \left( \frac{3}{2} \right) \max_{w \in [a + c, b + d]} |f_-^1(w)|. \)

Then the right focal problem (1), (2) has a solution \( x^* \in P \).

**Proof.** For \( x \in P \) define \( R \) and \( S \) on \( P \) by

\[ Rx = \int_0^1 G(t, \tau) f_+(x(\tau)) d\tau \quad \text{and} \quad Sx = \int_0^1 G(t, \tau) f_-(x(\tau)) d\tau. \]
We have that $R$ is increasing on $P$ since, if $x, y \in P$ with $x \leq y$, then for all $\tau \in [0, 1]$

$$f_+(x(\tau)) \leq f_+(y(\tau))$$

since $f_+$ is increasing, hence for all $t \in [0, 1]$

$$(Rx)(t) = \int_0^1 G(t, \tau) f_+(x(\tau)) \ d\tau \leq \int_0^1 G(t, \tau) f_+(y(\tau)) \ d\tau = (Ry)(t),$$

and therefore

$$Rx \leq Ry.$$  

Similarly we have that $S$ is decreasing on $P$ since, if $x, y \in P$ with $x \leq y$, then for all $\tau \in [0, 1]$

$$f_-(x(\tau)) \geq f_-(y(\tau)),$$

hence for all $t \in [0, 1]$

$$(Sx)(t) = \int_0^1 G(t, \tau) f_-(x(\tau)) \ d\tau \geq \int_0^1 G(t, \tau) f_-(y(\tau)) \ d\tau = (Sy)(t),$$

and therefore

$$Sx \geq Sy.$$  

Also, $R$, $S$ and $T$ are completely continuous operators on $P$ by applying an Arzela-Ascoli Theorem argument. Now, let

$$Y = \overline{P(a, a, b, b)} = \{ x \in P : a \leq a(x) \ and \ b(x) \leq b \}$$

and

$$Z = \overline{P(c, c, d, d)} = \{ x \in P : c \leq a(x) \ and \ b(x) \leq d \}.$$  

Claim 1: If $r \in Y$ and $s \in Z$ then $R(r + s) \in Y$ and $S(r + s) \in Z$.

Let $r \in Y$ and $s \in Z$, thus for $\tau \in \left[ \frac{1}{4}, 1 \right]$ we have

$$a + c \leq r(\tau) + s(\tau),$$

and for $\nu \in [0, 1]$, we have that

$$r(\nu) + s(\nu) \leq b + d.$$  

Thus we have that

$$a(R(r + s)) = \int_0^1 G \left( \frac{1}{4}, \tau \right) f_+(r(\tau) + s(\tau)) \ d\tau$$

$$\geq \frac{1}{4} \int_0^1 f_+(a + c) \ d\tau$$

$$= \frac{3f_+(a + c)}{16} \geq a,$$

and
\[ \beta(R(r + s)) = \int_0^1 G(1, v) \ f_+(r(v) + s(v)) \ dv \]
\[ \leq \int_0^1 v \ f_+(b + d) \ dv \]
\[ = \frac{f_+(b + d)}{2} \leq b. \]

Similarly we have that
\[ \alpha(S(r + s)) = \int_0^1 G(1, \nu) \ f_-(r(\nu) + s(\nu)) \ d\nu \]
\[ \geq \int_0^{1/4} \nu \ f_-(b + d) \ d\nu + \frac{1}{4} \int_{1/4}^1 f_-(b + d) \ d\nu \]
\[ = \frac{7f_-(b + d)}{32} > c \]

and
\[ \beta(S(r + s)) = \int_0^1 G(1, v) \ f_-(r(v) + s(v)) \ dv \]
\[ \leq \int_0^1 v \ f_-(0) \ dv \]
\[ = \frac{f_-(0)}{2} \leq d. \]

Therefore \( R(r + s) \in Y \) and \( S(r + s) \in Z \).

Let
\[ r_0 = \begin{cases} 
4at & \text{if } t \leq \frac{1}{4} \\
4ct & \text{if } t \leq \frac{1}{2} \\
4at & \text{if } t \geq \frac{1}{4} \\
\end{cases} \quad \text{and} \quad s_0 = \begin{cases} 
4at & \text{if } t \leq \frac{1}{4} \\
c & \text{if } t \geq \frac{1}{4} \end{cases} \]

thus
\[ r_0 \in Y \] and \( s_0 \in Z \).

For \( n \geq 1 \), define
\[ r_n = R(r_{n-1} + s_{n-1}) \quad \text{and} \quad s_n = S(r_{n-1} + s_{n-1}), \]

and for \( n \geq 0 \), define
\[ x_n = r_n + s_n. \]
We will verify that 
\[ r_n - r_{n-1} \geq s_{n-1} - s_n \]
for \( n \geq 1 \) by verifying that \( x_n \geq x_{n-1} \) in Claims 2 and 3 below.

Claim 2: \( x_1 \geq x_0 \) and \( x_1 - x_0 \) is an increasing function on \([0, 1] \).

For \( t < \frac{1}{4} \),

\[
x_1'(t) - x_0'(t) = \int_{t}^{1} f_+(r_0(\tau) + s_0(\tau)) \, d\tau + \int_{t}^{1} f_-(r_0(\tau) + s_0(\tau)) \, d\tau - 4a - 4c \\
\geq \int_{t}^{1} f_+(r_0(\tau) + s_0(\tau)) \, d\tau + \int_{t}^{1} f_-(r_0(\tau) + s_0(\tau)) \, d\tau - 4a - 4c \\
\geq \int_{t}^{1} f_+(a + c) \, d\tau + \int_{t}^{1} f_-(b + d) \, d\tau - 4a - 4c \geq 0
\]
since \( f_+(a + c) \geq \frac{16a}{3} \) and \( f_-(b + d) \geq \frac{16c}{3} \). For \( t > \frac{1}{4} \),

\[
x_1'(t) - x_0'(t) = \int_{t}^{1} f_+(r_0(\tau) + s_0(\tau)) \, d\tau + \int_{t}^{1} f_-(r_0(\tau) + s_0(\tau)) \, d\tau \geq 0
\]
since \( f_+ \) and \( f_- \) are non-negative functions. Consequently \((x_1 - x_0)(t)\) is an increasing function of \( t \) on the interval \([0, 1]\) and \( x_1(0) = x_0(0) = 0 \), thus we have that \( x_1 \geq x_0 \).

Claim 3: For \( n \geq 1 \), \( x_n \geq x_{n-1} \) and \( x_n - x_{n-1} \) is an increasing function on \([0, 1]\).

We proceed by induction. In Claim 2 it is shown the claim is true when \( n = 1 \), so suppose the claim is true for \( n = k \). For \( \tau \in [0, 1] \), if \( x_k(\tau) \neq x_{k-1}(\tau) \) by the mean value theorem let \( \zeta(\tau) \) and \( \phi(\tau) \) be elements of \([x_{k-1}(\tau), x_k(\tau)]\) such that

\[
f'_+(\zeta(\tau)) = \frac{f_+(x_k(\tau)) - f_+(x_{k-1}(\tau))}{x_k(\tau) - x_{k-1}(\tau)}
\]
and

\[
f'_-(\phi(\tau)) = \frac{f_-(x_k(\tau)) - f_-(x_{k-1}(\tau))}{x_k(\tau) - x_{k-1}(\tau)},
\]
and if \( x_k(\tau) = x_{k-1}(\tau) \) let \( \zeta(\tau) = \phi(\tau) = x_k(\tau) = x_{k-1}(\tau) \).

For \( t < \frac{1}{4} \),
\[ x_{k+1}(t) - x_k(t) = \int_1^t f_+(x(t)) + f_-(x(t)) \, d\tau \\
- \int_1^t f_+(x_{k-1}(\tau)) + f_-(x_{k-1}(\tau)) \, d\tau \\
= \int_1^t f_+(x_k(\tau)) - f_+(x_{k-1}(\tau)) \, d\tau \\
+ \int_1^t f_-(x_k(\tau)) - f_-(x_{k-1}(\tau)) \, d\tau \\
\geq \int_1^t \min_{x \in [a+c,b+d]} f_+(z)(x_k(\tau) - x_{k-1}(\tau)) \, d\tau \\
+ \int_1^t f_-(\phi(\tau))(x_k(\tau) - x_{k-1}(\tau)) \, d\tau \\
\geq \int_1^t \min_{x \in [a+c,b+d]} 2f_+(z)/3(x_k(\tau) - x_{k-1}(\tau)) \, d\tau \\
- \int_1^t \max_{w \in [a+c,b+d]} |f_-(w)|(x_k(\tau) - x_{k-1}(\tau)) \, d\tau \\
+ \int_1^t \left( \min_{x \in [a+c,b+d]} f_+(z) \right) (x_k(\tau) - x_{k-1}(\tau)) \, d\tau \\
+ \int_1^t f_-(\phi(\tau))(x_k(\tau) - x_{k-1}(\tau)) \, d\tau \\
\geq \min_{x \in [a+c,b+d]} f_+(z)/4 \left( x_k \left( \frac{1}{4} \right) - x_{k-1} \left( \frac{1}{4} \right) \right) \\
- \max_{w \in [0,b+d]} |f_-(w)|/4 \left( x_k \left( \frac{1}{4} \right) - x_{k-1} \left( \frac{1}{4} \right) \right) \\
\geq 0.
\]

Similarly for \( t > \frac{1}{4} \) we have that
\[ x_{k+1}(t) - x_k(t) \geq 0, \]
making \((x_{k+1} - x_k)(t)\) an increasing function of \( t \) on the interval \([0,1]\) and \( x_{k+1}(0) = x_k(0) = 0 \), thus we have that \( x_{k+1} \geq x_k \) which completes our induction argument.

Therefore by Theorem 2.2 the sequence \( r_n \) increases to \( r^* \), the sequence \( s_n \) decreases to \( s^* \), and
\[ x_n = r_n + s_n \rightarrow r^* + s^* = x^* \]
with \( x_n \) increasing to \( x^* \) which is a solution in \( Y + Z \) of our boundary value problem (1), (2). 

\[ \square \]
References


