Normalized Prepared Bases for Discrete Symplectic Matrix Systems

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Abstract

We will find conditions on one pair of a normalized prepared basis of a discrete symplectic matrix system that lead to the other pair being a dominant solution at $\infty$. The characterization of a recessive solution of the symplectic system at $\infty$ will also be given.

Key words: discrete symplectic matrix system, prepared solution, prepared basis, dominant and recessive solutions
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1 Introduction

Consider the discrete symplectic matrix system

$$
Y(t + 1) = E(t)Y(t) + F(t)Z(t)
$$

$$
Z(t + 1) = G(t)Y(t) + H(t)Z(t)
$$

on the discrete interval $[a, \infty)$, where all matrices are $n \times n$, and the coefficient matrix

$$
\begin{bmatrix}
E(t) & F(t) \\
G(t) & H(t)
\end{bmatrix}
$$

is symplectic. Recall that a $2n \times 2n$ matrix $M$ is symplectic provided

$$
M^*JM = J,
$$

where $*$ denotes the conjugate transpose of the matrix, and where the $2n \times 2n$ constant matrix $J$ is given by

$$
J = \begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}.
$$

Ahlbrandt and Peterson [1, page 83] show that (1) is equivalent to a linear matrix Hamiltonian system if $E(t)$ is nonsingular for all $t \geq a$. Although we will not formally require $F(t)$ to also be nonsingular for $t \geq a$, it turns out that the results of the paper hold actually only for certain hypotheses that force $F(t)$ to be invertible.

**Definition 1** For any $n \times n$ matrix pairs of solutions $Y_1(t), Z_1(t)$ and $Y_2(t), Z_2(t)$ of (1), define

$$
\left\{ \begin{array}{cc}
Y_1(t) & Y_2(t) \\
Z_1(t) & Z_2(t)
\end{array} \right\} := Y_1^*(t)Z_2(t) - Z_1^*(t)Y_2(t).
$$

The following lemmas and theorems in this introductory section are all proven in Ahlbrandt and Peterson [1]. Some related papers include Anderson [2], Bohner [3], and Bohner and Došíly [4].

**Lemma 1** If $Y_1(t), Z_1(t)$ and $Y_2(t), Z_2(t)$ are $n \times n$ solutions of (1), then for $t \geq a$

$$
\left\{ \begin{array}{cc}
Y_1(t) & Y_2(t) \\
Z_1(t) & Z_2(t)
\end{array} \right\} = A,
$$

where $A$ is an $n \times n$ constant matrix.

**Definition 2** If

$$
\left\{ \begin{array}{cc}
Y(t) & Y(t) \\
Z(t) & Z(t)
\end{array} \right\} = 0,
$$

then $Y, Z$ is said to be a prepared solution of (1). If, in addition,

$$
\text{rank} \left[ \begin{array}{c}
Y(t) \\
Z(t)
\end{array} \right] = n,
$$

then we say $Y, Z$ is a prepared basis of (1). If $Y_i(t), Z_i(t)$ are prepared bases of (1) for $i = 1, 2$ such that

$$
\left\{ \begin{array}{cc}
Y_1(t) & Y_2(t) \\
Z_1(t) & Z_2(t)
\end{array} \right\} = I,
$$

then $Y_1, Z_1$ and $Y_2, Z_2$ are normalized prepared bases of solutions of (1).

**Theorem 1** If $Y_1, Z_1$ and $Y_2, Z_2$ are normalized prepared bases for (1), we have the following:

$$
Y_1(t)Y_2^*(t) = Y_2(t)Y_1^*(t), \quad (2)
$$

$$
Z_1(t)Z_2^*(t) = Z_2(t)Z_1^*(t), \quad (3)
$$

$$
Z_2(t)Y_1^*(t) - Z_1(t)Y_2^*(t) = I. \quad (4)
$$

$$
Y_2(t + 1)Y_1^*(t) - Y_1(t + 1)Y_2^*(t) = F(t). \quad (5)
$$

The following theorem is proven in Ahlbrandt and Peterson [1, page 121].
Theorem 2 (Normal Basis Theorem) Assume $Y_1(t), Z_1(t)$ and $Y_2(t), Z_2(t)$ are normal pairs of solutions of (1). If $\Gamma_1$ and $\Gamma_2$ are $n \times n$ constant matrices, then
\[
Y(t) = Y_1(t)\Gamma_1 + Y_2(t)\Gamma_2
\]
\[
Z(t) = Z_1(t)\Gamma_1 + Z_2(t)\Gamma_2
\]
is a solution of (1), where
\[
\Gamma_1 = -\begin{bmatrix} Y_2(t) & Y(t) \\ Z_2(t) & Z(t) \end{bmatrix}, \Gamma_2 = \begin{bmatrix} Y_1(t) & Y(t) \\ Z_1(t) & Z(t) \end{bmatrix},
\]
and
\[
\begin{bmatrix} Y(t) & Y(t) \\ Z(t) & Z(t) \end{bmatrix} = \Gamma_1^*\Gamma_2 - \Gamma_2^*\Gamma_1.
\]

Conversely, if $Y(t), Z(t)$ is a solution of (1), then $Y(t)$ and $Z(t)$ are given by (6) and (7) when $\Gamma_1, \Gamma_2$ are defined by (8).

Theorem 3 (Reduction of Order) If $Y_0(t), Z_0(t)$ is a prepared $n \times n$ matrix solution of (1) such that $Y_0(t)$ is invertible on $[a, b]$ and
\[
Y(t) = Y_0(t) \left[ P + \sum_{s=a}^{t-1} Y_0^{-1}(s+1)F(s)Y_0^{-1}(s)Q \right]
\]
\[
Z(t) = Z_0(t) \left[ P + \sum_{s=a}^{t-1} Y_0^{-1}(s+1)F(s)Y_0^{-1}(s)Q \right] + Y_0^{*^{-1}}(t)Q
\]
for $t \in [a, b]$ and constant matrices $P, Q$, then $Y(t), Z(t)$ is a solution of (1), where
\[
P = Y_0^{-1}(a)Y(a), \quad Q = \begin{bmatrix} Y_0(t) & Y(t) \\ Z_0(t) & Z(t) \end{bmatrix}.
\]

2 Main Results

Definition 3 Let $Y, Z$ be a prepared solution of (1). We say $Y(t)$ has a generalized zero at $a$ if $Y(a)$ is singular. Otherwise, $Y(t)$ has a generalized zero at $t_0$ for $t_0 > a$ provided $\det Y(t_0 - 1) \neq 0$ and either $Y(t_0)$ is singular or
\[
Y^{-1}(t_0)F(t_0 - 1)Y^{*^{-1}}(t_0 - 1)
\]
is nonsingular and not positive definite. In other words, if $Y(t)$ does not have a generalized zero at $t_0$ and $Y(t_0 - 1)$ is nonsingular, then
\[
Y^{-1}(t_0)F(t_0 - 1)Y^{*^{-1}}(t_0 - 1) > 0.
\]
(Here the inequality $A > 0$ signifies that $A$ is an $n \times n$ positive definite Hermitian matrix.)
Theorem 4 Let $Y_1, Z_1$ and $Y_2, Z_2$ be normalized prepared bases of (1) with $Y_2(a) = 0$. If $Y_1(t)$ has no generalized zeros on $[a, b]$, then $Y_2(t)$ has no generalized zeros on $[a + 1, b]$.

Proof: Since $Y_1(t)$ has no generalized zeros on $[a, b]$,

$$Y_1^{-1}(t + 1)F(t)Y_1^{*-1}(t) > 0$$

for $t \in [a, b - 1]$; in particular, $F(t)$ and $Y_1(t)$ are nonsingular on $[a, b]$. By the Reduction of Order Theorem (Theorem 3),

$$Y_2(t) = Y_1(t) \sum_{s=a}^{t-1} Y_1^{-1}(s + 1)F(s)Y_1^{*-1}(s) \quad (10)$$

on $[a, b]$. For notational purposes, set

$$S(t) := \sum_{s=a}^{t-1} Y_1^{-1}(s + 1)F(s)Y_1^{*-1}(s); \quad (11)$$

by (2) and (10),

$$S(t) = Y_1^{-1}(t)Y_2(t)$$

and $S^*(t) = S(t)$ for all $t \in [a, b]$. It follows that

$$Y_2^*(t)F^{-1}(t)Y_2(t + 1) = S(t)Y_1^*(t)F^{-1}(t)Y_1(t + 1)S(t + 1) = S(t)Y_1^*(t)F^{-1}(t)Y_1(t + 1) + [Y_1^{-1}(t + 1)F(t)Y_1^{*-1}(t)]F(t)Y_1^{*-1}(t) + S(t)] = S(t) + S(t)Y_1^*(t)F^{-1}(t)Y_1(t + 1)S(t) = S(t) + S^*(t) [Y_1^{-1}(t + 1)F(t)Y_1^{*-1}(t)]^{-1} S(t).$$

Hence $Y_2^*(t)F^{-1}(t)Y_2(t + 1) > 0$ for $t \in [a + 1, b - 1]$, so that

$$Y_2^{-1}(t + 1)F(t)Y_2^{*-1}(t) > 0$$

on $[a + 1, b - 1]$; i.e., $Y_2(t)$ has no generalized zeros on $[a + 2, b]$. Since $Y_2(a) = 0$ and

$$Y_2(a + 1) = F(a)Y_1^{*-1}(a)$$

is invertible, $Y_2(t)$ has no generalized zero at $t = a + 1$ as well. \hfill \Box

Theorem 5 Let $Y_1, Z_1$ and $Y_2, Z_2$ be normalized prepared bases of (1). If $Y_1(t)$ has no generalized zeros on $[t_0, t_1]$ for any $t_0 \geq a$ with $t_1 > t_0 + n$, then $Y_2(t)$ has at most $n$ singularities on $[t_0, t_1]$. Likewise if $Y_2(t)$ has no generalized zeros on $[t_0, t_1]$ for $t_0 > a$, then $Y_1(t)$ has at most $n$ singularities on $[t_0, t_1]$. 

Proof: Let \( \alpha, \beta \in [t_0, t_1] \) such that \( \beta > \alpha \). By (11),

\[
\Delta S(t) = Y^{-1}_1(t + 1)F(t)Y^{-1}_1(t) > 0;
\]

thus

\[
S(\beta) > S(\alpha). \tag{12}
\]

For each \( t \in [t_0, t_1] \) let \( \lambda_i(t) \) be the \( i \)th eigenvalue of \( S(t) \); the functions \( \lambda_i(t) \) are increasing over \([t_0, t_1] \) by (12). The singularities of \( Y_2(t) \) are precisely those of \( S(t) = Y^{-1}_1(t)Y_2(t) \), and since \( S(t) \) is Hermitian, the singularities of \( S(t) \) occur at the zeros of the \( \lambda_i(t) \). Since the \( \lambda_i(t) \) are increasing for all \( i = 1, \ldots, n \), \( Y_2(t, a) \) has at most \( n \) singularities in \([t_0, t_1] \).

Now suppose \( Y_2(t) \) has no generalized zeros on \([t_0, t_1] \) for \( t > a \). Similar to the discussion leading to (10), the Reduction of Order Theorem gives that

\[
Y^{-1}_2(t)Y_1(t) = - \sum_{s=0}^{t-1} \left[ Y^{-1}_2(s + 1)F(s)Y^{-1}_2(s) \right] + Y^{-1}_2(t_0)Y_1(t_0) \tag{13}
\]

for \( t \in [t_0, t_1] \). Consequently, \( Y^{-1}_2(t)Y_1(t) \) is decreasing on \([t_0, t_1] \), and the result for \( Y_1(t) \) follows with reasoning similar to that above. \( \square \)

Notation: For Theorems 6 and 7 and Corollary 1, let \( Y_1(t, t_0), Z_1(t, t_0) \) and \( Y_2(t, t_0), Z_2(t, t_0) \) be the solutions of (1) determined by the initial conditions

\[
Y_1(t_0, t_0) = I \quad \text{and} \quad Y_2(t_0, t_0) = 0
\]

\[
Z_1(t_0, t_0) = 0 \quad \text{and} \quad Z_2(t_0, t_0) = I. \tag{14}
\]

Note that \( Y_1(t, t_0), Z_1(t, t_0) \) and \( Y_2(t, t_0), Z_2(t, t_0) \) constitute normalized prepared bases by the initial conditions given in (14) and by Lemma 1.

Theorem 6 If there is a \( t_0 > a \) such that \( Z_1(t_0, a) \) is nonsingular, and if there is a \( t_1 > t_0 + n \) such that \( Y_1(t, t_0) \) has no generalized zeros on \([t_0, t_1] \), then \( Y_1(t, a) \) has at most \( n \) singularities on \([t_0, t_1] \).

Proof: Since \( Y_1(t, t_0) \) has no generalized zeros on \([t_0, t_1] \), \( Y_1(t, t_0) \) is nonsingular on \([t_0, t_1] \). Let

\[
A(t) := Z_1^*(t_0, a)Y^{-1}_1(t, t_0)Y_1(t, a)
\]

for \( t \in [t_0, t_1] \). Since

\[
Y_1(t, a) = Y_1(t, t_0)Y_1(t_0, a) + Y_2(t, t_0)Z_1(t_0, a) \tag{15}
\]

by the Normal Basis Theorem,

\[
A(t) = Z_1^*(t_0, a)[Y_1(t_0, a) + Y^{-1}_1(t, t_0)Y_2(t, t_0)Z_1(t_0, a)]
\]

\[
= Z_1^*(t_0, a)Y_1(t_0, a) + Z_1^*(t_0, a)Y^{-1}_1(t, t_0)Y_2(t, t_0)Z_1(t_0, a).
\]

Then the difference of both sides produces

\[
\Delta A(t) = Z_1^*(t_0, a)\Delta \left[ Y^{-1}_1(t, t_0)Y_2(t, t_0) \right] Z_1(t_0, a).
\]
By (10),
\[ \Delta A(t) = Z_1^*(t_0, a) \left[ Y_1^{-1}(t + 1, t_0) F(t) Y_1^{* -1}(t, t_0) \right] Z_1(t_0, a), \]
so that \( \Delta A(t) > 0 \) on \([t_0, t_1 - 1]\), since \( Z_1(t_0, a) \) is nonsingular and
\[ Y_1^{-1}(t + 1, t_0) F(t) Y_1^{* -1}(t, t_0) > 0 \]
on \([t_0, t_1 - 1]\) by hypothesis. Hence \( A(t) \) is increasing on \([t_0, t_1]\), and as in the proof of Theorem 5, the functions \( \lambda_i(t) \), where \( \lambda_i(t) \) is the \( i \)th eigenvalue of \( A(t) \), are increasing on \([t_0, t_1]\). By the definition of \( A(t) \), the zeros of these functions are exactly the singularities of \( Y_1(t, a) \). Consequently, \( Y_1(t, a) \) has at most \( n \) singularities on \([t_0, t_1]\).

\( \square \)

**Definition 4** An \( n \times n \) prepared solution of (1) is said to be a dominant solution at \( \infty \) provided there is an integer \( t_0 \in [a, \infty) \) such that \( Y(t) \) in invertible on \([t_0, \infty)\) and
\[ \sum_{s=t_0}^{\infty} Y^{-1}(s + 1) F(s) Y^{* -1}(s) \]
converges to a Hermitian matrix with finite entries. A solution \( Y_0(t), Z_0(t) \) is a recessive solution at \( \infty \) for (1) provided \( Y_0(t), Z_0(t) \) is a prepared basis of (1) and provided that whenever \( Y(t), Z(t) \) is another solution of (1) with
\[ K = \begin{pmatrix} Y_0(t) & Y(t) \\ Z_0(t) & Z(t) \end{pmatrix} \]
a nonsingular \( n \times n \) constant matrix, it follows that there is a \( t_0 \geq a \) such that \( Y(t) \) is nonsingular for \( t \geq t_0 \) and
\[ \lim_{t \to \infty} Y^{-1}(t) Y_0(t) = 0. \]

**Theorem 7** If there exists \( t_0 \geq a \) such that \( Y_1(t, a) \) has no generalized zeros on \([t_0, \infty)\), then

(i) \( Y_2(t, t_0) \) has no generalized zeros on \([t_0 + 1, \infty)\);

(ii) \( Y_2(t, t_0), Z_2(t, t_0) \) is a dominant solution of (1) at \( \infty \);

(iii) the pair
\[ Y_0(t) := Y_2(t, t_0) \sum_{s=t}^{\infty} Y_2^{-1}(s + 1, t_0) F(s) Y_2^{* -1}(s, t_0) \]
\[ Z_0(t) := Z_2(t, t_0) \sum_{s=t}^{\infty} [Y_2^{-1}(s + 1, t_0) F(s) Y_2^{* -1}(s, t_0)] - Y_2^{* -1}(t, t_0) \]
is a well-defined recessive solution of (1) at \( \infty \).
Proof: Set
\[ Y_1(t) \equiv Y_1(t, a) \quad \text{and} \quad Z_1(t) \equiv Z_1(t, a). \]
Since \( Y_1(t) \) has no generalized zeros on \([t_0, \infty)\), \( Y_1(t) \) is invertible on \([t_0 - 1, \infty)\), and
\[ Y_1^{-1}(t + 1)F(t)Y_1^{-1}(t) > 0 \]
on \([t_0 - 1, \infty)\); this implies that \( F(t) \) is nonsingular for \( t \in [t_0 - 1, \infty) \) as well. Let
\[ S(t) := \sum_{s=t_0}^{t-1} Y_1^{-1}(s + 1)F(s)Y_1^{-1}(s); \] (16)
then \( S(t) > 0 \) for \( t \geq t_0 + 1 \). By the Reduction of Order Theorem,
\[ Y_2(t, t_0) = Y_1(t)S(t)Y_1^*(t_0) \] (17)
for \( t \in [t_0, \infty) \). Thus, we see that
\[ Y_2^*(t, t_0)F^{-1}(t)Y_2(t + 1, t_0) = Y_1(t_0)S(t)Y_1^*(t)F^{-1}(t)Y_1(t + 1)S(t + 1)Y_1^*(t_0) \]
\[ = Y_1(t_0)S(t)Y_1^*(t)F^{-1}(t)Y_1(t + 1) \]
\[ \cdot \left[ Y_1^{-1}(t + 1)F(t)Y_1^{-1}(t) + S(t) \right] Y_1^*(t_0) \]
\[ = Y_1(t_0)S(t)[Y_1^{-1}(t + 1)F(t)Y_1^{-1}(t)]^{-1}S(t)Y_1^*(t_0) \]
\[ + Y_1(t_0)S(t)Y_1^*(t_0) \]
\[ > 0 \]
on \([t_0 + 1, \infty)\), since \( Y_1(t_0) \) is nonsingular and \( S(t) > 0 \) for \( t \geq t_0 \). As a result,
\[ Y_2^{-1}(t + 1, t_0)F(t)Y_2^{-1}(t, t_0) > 0 \]
on \([t_0 + 1, \infty)\), so that \( Y_2(t, t_0) \) has no generalized zeros on \([t_0 + 2, \infty)\). Since \( Y_2(t_0, t_0) = 0 \), and \( Y_2(t_0 + 1, t_0) = F(t_0) \) is invertible, \( Y_2(t, t_0) \) has no generalized zeros on \([t_0 + 1, \infty)\), and (i) is established. For notational purposes, for the remainder of the proof set
\[ X(t) := Y_2^{-1}(t + 1, t_0)F(t)Y_2^{-1}(t, t_0). \] (18)
By (16) and (17),
\[ 0 < Y_1^*(t_0)Y_2^{-1}(t, t_0)Y_1(t). \] (19)
From (15) we have
\[ Y_1(t) = Y_1(t, t_0)Y_1(t_0) + Y_2(t, t_0)Z_1(t_0). \]
Consequently, we rewrite (19) and get
\[ 0 < Y_1^*(t_0)[Y_2^{-1}(t, t_0)Y_1(t, t_0)Y_1(t_0) + Z_1(t_0)] \]
\[ = Y_1^*(t_0)Y_2^{-1}(t, t_0)Y_1(t, t_0)Y_1(t_0) + Y_1^*(t_0)Z_1(t_0). \]
Using (13) and (19) we have
\[ 0 < Y_1^*(t_0) \left[ Y_2^{-1}(t + 1, t_0)Y_1(t_0 + 1, t_0) - \sum_{s=t_0+1}^{t-1} X(s) \right] Y_1(t_0) \]
\[ + Y_1^*(t_0)Z_1(t_0). \]
This leads to
\[ Y_1^*(t_0) \sum_{s=t_0+1}^{t-1} X(s)Y_1(t_0) < Y_1^*(t_0) \left[ Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0)Z_1(t_0)Y_1^{-1}(t_0) \right] Y_1(t_0) \]
for \( t \geq t_0 + 1 \); thus,
\[ 0 < \sum_{s=t_0+1}^{t-1} X(s) < Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0) + Z_1(t_0)Y_1^{-1}(t_0) \]
for \( t \geq t_0 + 2 \). Therefore,
\[ 0 < \sum_{s=t_0+1}^{t-1} \text{tr} X(s) < \text{constant} < \infty \]
for all \( t \geq t_0 + 2 \), where \( \text{tr} X(s) \) is the trace of the matrix \( X(s) \). It follows that
\[ 0 < \sum_{s=t_0+1}^{\infty} \text{tr} X(s) < \infty. \] (20)

Let
\[ [x_{ij}] := X(t) = Y_2^{-1}(t + 1, t_0)F(t)Y_2^{-1}(t, t_0); \]
we have \( X(t) > 0 \) and
\[ \|X(t)\|_2 < \text{tr} X(t). \] (21)
As a result,
\[ \sum_{t=t_0+1}^{\infty} |x_{ij}(t)| = \sum_{t=t_0+1}^{\infty} \sqrt{|x_{ij}(t)|^2} \leq \sum_{t=t_0+1}^{\infty} \sqrt{\sum_{i,j=1}^{n} |x_{ij}(t)|^2} \]
\[ = \sum_{t=t_0+1}^{\infty} \|X(t)\|_2 \leq \sum_{t=t_0+1}^{\infty} \text{tr} X(t) \]
\[ < \infty. \]

Therefore
\[ \sum_{t=t_0+1}^{\infty} X(t) = \sum_{t=t_0+1}^{\infty} Y_2^{-1}(t + 1, t_0)F(t)Y_2^{-1}(t, t_0) \]
exists and is Hermitian. Using the fact that \( Y_2(t, t_0), Z_2(t, t_0) \) is a prepared solution of (1), we have that \( Y_2(t, t_0), Z_2(t, t_0) \) is a dominant solution of (1) at \( \infty \), which is (ii). We can thus write
\[ Y_2^{-1}(t, t_0)Y_1(t, t_0) - Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0) = \sum_{s=t}^{\infty} X(s) - \sum_{s=t_0+1}^{\infty} X(s) \] (22)
using the Reduction of Order Theorem. Rewrite (22) as
\[ Y_2(t, t_0) \sum_{s=t_0+1}^{\infty} X(s) + Y_1(t, t_0) - Y_2(t, t_0)Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0) = Y_2(t, t_0) \sum_{s=t}^{\infty} X(s); \]
if
\[ \hat{Y}_0(t) := Y_1(t, t_0) + Y_2(t, t_0) \sum_{s=t_0+1}^{\infty} X(s) - Y_2(t, t_0)Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0) \]
and
\[ Y_0(t) := Y_2(t, t_0) \sum_{s=t}^{\infty} X(s), \]
then \( \hat{Y}_0(t) = Y_0(t) \) for \( t \geq t_0 + 1 \). Similarly, by (2) and (22) we have
\[ Y_1^*(t, t_0)Y_2^{s-1}(t, t_0) - Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0) = \sum_{s=t}^{\infty} X(s) - \sum_{s=t_0+1}^{\infty} X(s), \]
so that
\[
Z_2(t, t_0) \sum_{s=t_0+1}^{\infty} X(s) - Z_2(t, t_0)Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0)
= Z_2(t, t_0) \sum_{s=t}^{\infty} X(s) - Z_2(t, t_0)Y_1^*(t, t_0)Y_2^{s-1}(t, t_0).
\]
Now use (4) to arrive at
\[
Z_2(t, t_0) \sum_{s=t_0+1}^{\infty} X(s) - Z_2(t, t_0)Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0) + Z_1(t, t_0)
= Z_2(t, t_0) \sum_{s=t}^{\infty} X(s) - Y_2^{s-1}(t, t_0).
\]
Hence, if
\[ \hat{Z}_0(t) := Z_1(t, t_0) + Z_2(t, t_0) \sum_{s=t_0+1}^{\infty} X(s) - Z_2(t, t_0)Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0) \]
and
\[ Z_0(t) := Z_2(t, t_0) \sum_{s=t}^{\infty} X(s) - Y_2^{s-1}(t, t_0), \]
then \( \hat{Z}_0(t) = Z_0(t) \) for \( t \geq t_0 + 1 \). By the Normal Basis Theorem, \( \hat{Y}_0(t), \hat{Z}_0(t) \) is a solution of (1) on \([t_0, \infty)\), with \( \Gamma_1 = I \) and \( \Gamma_2 = \sum_{s=t_0+1}^{\infty} X(s) - Y_2^{-1}(t_0 + 1, t_0)Y_1(t_0 + 1, t_0) \). Consequently \( Y_0(t), Z_0(t) \) given by (23) and (24), respectively, is a solution of (1) on \([t_0 + 1, \infty)\). The proof that a solution of the form of our \( Y_0, Z_0 \) is a recessive solution of (1) at \( \infty \) can be found in Ahlbrandt and Peterson [1, pages 128-130].

**Corollary 1** For the same hypothesis on \( Y_1(t, a) \) given in Theorem 7,
\[
\sum_{t=t_0+1}^{\infty} tr F(t)\|Y_2(t, t_0)\|^{-1}\|Y_2(t + 1, t_0)\|^{-1} < \infty.
\]
Proof: Again let $Y_1(t) \equiv Y_1(t,a)$ for all $t \geq a$. Recall that under this hypothesis, we have

$$
\sum_{t=t_0+1}^{\infty} \text{tr} \left( Y_2^{-1}(t+1, t_0) F(t) Y_2^{*-1}(t, t_0) \right) < \infty
$$

from (20), and

$$
\|Y_2^{-1}(t+1, t_0) F(t) Y_2^{*-1}(t, t_0)\| \leq \text{tr} \left( Y_2^{-1}(t+1, t_0) F(t) Y_2^{*-1}(t, t_0) \right)
$$

from (21). Note that

$$
\|F(t)\| = \|Y_2(t+1, t_0) Y_2^{-1}(t+1, t_0) F(t) Y_2^{*-1}(t, t_0) Y_2^{*}(t, t_0)\| \\
\leq \|Y_2(t+1, t_0)\| \cdot \|Y_2^{-1}(t+1, t_0) F(t) Y_2^{*-1}(t, t_0)\| \cdot \|Y_2^{*}(t, t_0)\|.
$$

Using the fact that $\|A\| \geq \frac{1}{n} \text{tr} A$, together with (26) and (27), we obtain the following:

$$
\text{tr} \left( Y_2^{-1}(t+1, t_0) F(t) Y_2^{*-1}(t, t_0) Y_2^{*}(t, t_0) \right) \geq \|F(t)\| \cdot \|Y_2(t+1, t_0)\|^{-1} \cdot \|Y_2^{*-1}(t, t_0)\|^{-1} \\
\geq \frac{1}{n} \text{tr} F(t) \|Y_2(t+1, t_0)\|^{-1} \|Y_2^{*-1}(t, t_0)\|^{-1}.
$$

Summation and (25) result in

$$
\infty \geq n \sum_{t=t_0+1}^{\infty} \text{tr} \left( Y_2^{-1}(t+1, t_0) F(t) Y_2^{*-1}(t, t_0) \right) \\
\geq \sum_{t=t_0+1}^{\infty} \text{tr} F(t) \|Y_2(t+1, t_0)\|^{-1} \|Y_2^{*-1}(t, t_0)\|^{-1}.
$$

\[\square\]

References


