Hyers–Ulam Stability for a Continuous Time Scale with Discrete Uniform Jumps

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Dedicated to Johnny Henderson on the occasion of his 70th birthday.

Abstract

We investigate the Hyers–Ulam stability (HUS) of a certain first-order linear complex constant coefficient dynamic equation on the time scale \( \mathbb{P}_{\alpha,h} \), which has continuous intervals of length \( \alpha > 0 \) followed by discrete jumps of length \( h > 0 \). In particular, we establish results in the case of this specific time scale, for co-efficient values in the complex plane, including where the exponential function alternates in sign. In our analysis, we employ the Lambert \( W \) function. For increasing jump size \( h \) relative to \( \alpha \), we prove that the complex constant coefficient undergoes a bifurcation in its parameter space. We establish interesting results for both the delta dynamic equation and the nabla dynamic equation.

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1 Content Prelude

Imagine a data burst or transmission signal broadcast over a short time period, and then repeated, or a device or system that runs continuously for a fixed time, shuts off, and then runs again. In biology, think of an organism that lives a fixed unit of time, followed by hibernation or dormancy, and then is active again, and so on. These scenarios may be modeled by a specific time scale \( \mathbb{T} \) with fixed jump size that displays both continuous
and discrete properties, see Bohner and Peterson [8, Examples 1.38–1.40], where a time
scale is any closed subset of the real line \( \mathbb{R} \). In particular, let \( T = \mathbb{P}_{\alpha,h} \) for continuous
interval length \( \alpha > 0 \) and discrete jump size \( h > 0 \), namely
\[
\mathbb{P}_{\alpha,h} = \bigcup_{k=0}^{\infty} [k(\alpha + h), k(\alpha + h) + \alpha],
\]
and consider the differential operator defined by
\[
x^\Delta(t) = \begin{cases}
\frac{d}{dt} x(t) & \text{for } t \in [k(\alpha + h), k(\alpha + h) + \alpha) \\
\frac{x(t + h) - x(t)}{h} & \text{for } t = k(\alpha + h) + \alpha.
\end{cases}
\]

We will be investigating some stability questions for this time scale and this derivative
operator, in the Hyers-Ulam sense, defined below. Note that this problem is explored
briefly but incompletely in [2, Example 4.1]. Our aim is to give a more robust analysis
of the situation in this work.

Time scales were introduced by Hilger [12] to unify continuous and discrete anal-
ysis. Hyers–Ulam stability was initiated by Ulam [27], followed by Hyers [13] and
Rassias [25]. Some early work on differential equations and this type of stability in-
clude Miura et al [18, 19] and Jung et al [14–17]. Some recent papers on HUS and
difference equations or more generally time scales include Anderson [1], Anderson and
Onitsuka [2–6], Andras et al [7], Brzdek et al [9], Buse et al [10], Nam [20–22], Onit-
suka [23, 24], and Shen [26].

**Definition 1.1** (Hyers-Ulam Stability). We say that
\[
x^\Delta(t) = \lambda x(t), \quad \lambda \in \mathbb{C} \setminus \left\{ \frac{-1}{h} \right\}, \quad t \in T
\]
has Hyers–Ulam stability on \( T \) if and only if there exists a constant \( K > 0 \) with the
following property. For arbitrary \( \varepsilon > 0 \), if a function \( \phi : T \to \mathbb{C} \) satisfies
\[
|\phi^\Delta(t) - \lambda \phi(t)| \leq \varepsilon, \quad t \in T,
\]
then there exists a solution \( x : T \to \mathbb{C} \) of (1.1) such that \( |\phi(t) - x(t)| \leq K\varepsilon \) for all
\( t \in T \). Such a constant \( K \) is called an HUS constant for (1.1) on \( T \).

In this work, we consider the time scale \( T = \mathbb{P}_{\alpha,h} \), and the time scale eigenvalue
problem given in (1.1). For \( t \in T \), we have the forward jump operator \( \sigma \) defined by
\[
\sigma(t) := \begin{cases}
t & \text{for } t \in [k(\alpha + h), k(\alpha + h) + \alpha), \\
t + h & \text{for } t = k(\alpha + h) + \alpha.
\end{cases}
\]
For \( \lambda \in \mathbb{C} \setminus \left\{-\frac{1}{h}\right\} \), the exponential function \( e_\lambda(t, 0) \) is given by

\[
e_\lambda(t, 0) = (1 + h\lambda)^k e^{\lambda(t-hk)}, \quad t \in [k(\alpha + h), k(\alpha + h) + \alpha], \quad k \in \mathbb{N}_0, \tag{1.3}
\]

which can also be written as

\[
e_\lambda(t, 0) = \left[(1 + h\lambda)e^{\alpha \lambda}\right]^k e^{\lambda j}, \quad t = k(\alpha + h) + j, \quad j \in [0, \alpha].
\]

Clearly, the exponential function in (1.3) is well defined for \( \lambda \in \mathbb{C} \setminus \left\{-\frac{1}{h}\right\} \). Notice that

\[
x(t) = x_0 e_\lambda(t, 0), \quad t \in \mathbb{T}, \tag{1.4}
\]

is the general solution of (1.1), for the exponential function \( e_\lambda \) given in (1.3). Moreover, for given \( \varepsilon > 0 \), the function

\[
\phi(t) = \phi_0 e_\lambda(t, 0) + e_\lambda(t, 0) \int_0^t \frac{q(s)}{e_\lambda(\sigma(s), 0)} \Delta s, \quad |q(s)| \leq \varepsilon \forall s \in \mathbb{T}, \tag{1.5}
\]

where

\[
\int_{k(\alpha + h)}^t \frac{q(s)}{e_\lambda(\sigma(s), 0)} \Delta s = \int_{k(\alpha + h)}^t \frac{q(s)}{e_\lambda(s, 0)} ds, \quad t \in [k(\alpha + h), k(\alpha + h) + \alpha]
\]

and

\[
\int_{k(\alpha + h) - h}^{k(\alpha + h)} \frac{q(s)}{e_\lambda(\sigma(s), 0)} \Delta s = \frac{hq(k(\alpha + h) - h)}{e_\lambda(k(\alpha + h), 0)}
\]

is the solution of (1.2) by the variation of parameters formula, see Bohner and Peterson [8, Theorem 2.77].

Throughout the paper, we will need to employ the Lambert \( W \) function, see Corless et al [11], which we denote by \( W_z \), where \( W_z \) satisfies \( W_z(y)e^{W_z(y)} = y \), for every \( z \in \mathbb{Z} \). For example, using (1.3) and \( t = k(\alpha + h) \), we have

\[
e_\lambda(k(\alpha + h), 0) = \left[(1 + h\lambda)e^{\alpha \lambda}\right]^k.
\]

To prevent this from vanishing, we always assume \( \lambda \neq -\frac{1}{h} \). Moreover, we will see that other key values for \( \lambda \in \mathbb{R} \) include when the base \( (1 + h\lambda)e^{\alpha \lambda} = \pm 1 \). If \( \lambda = 0 \), then \( (1 + h\lambda)e^{\alpha \lambda} = 1 \), but, for \( \lambda \in \mathbb{R} \), we note here that for the branches \( z = -1, 0 \) of the Lambert \( W = W_z \) function,

\[
(1 + h\lambda)e^{\alpha \lambda} = -1 \iff \lambda = -\frac{1}{h} + \frac{1}{\alpha} W_0 \left(-\frac{\alpha}{h} e^{\frac{\alpha}{h}}\right) \quad \text{and} \quad h \geq \frac{\alpha}{W_0(e^{-1})} \approx 3.59112\alpha,
\]

where \( h > 0 \) is the jump size, and \( W_0 \) is the principal branch of the Lambert \( W \) function. In particular, if \( h = \frac{\alpha}{W_0(e^{-1})} \) and \( \lambda = -\frac{1}{h} - \frac{1}{\alpha} \approx -1.27846 \frac{\alpha}{\alpha} \), then \( e_\lambda(k(\alpha + h), 0) = (-1)^k \).
2 Hyers–Ulam Stability on $\mathbb{P}_{\alpha,h}$

We now give our first new results, when the eigenvalue $\lambda$ is a real number; in a later section, we will consider the more general case of $\lambda \in \mathbb{C}$. Moreover, we will fix $\alpha > 0$ and let the jump size $h > 0$ range over all positive real numbers in relation to $\alpha$. Of course, one could also fix $h > 0$ and let $\alpha > 0$ vary, as well. For the sake of completeness, we will include the details of proofs for this specific time scale $T = \mathbb{P}_{\alpha,h}$.

We will refer to the following constant, 

$$K_{\mathbb{R}} = \frac{1}{-\lambda} \left( \frac{e^{\alpha \lambda}(1 + h\lambda) - (1 + 2h\lambda)}{1 + e^{\alpha \lambda}(1 + h\lambda)} \right),$$

(2.1) throughout the remainder of this section.

**Theorem 2.1** (Delta equation). Fix $\alpha > 0$, and let $\lambda \in \mathbb{R} \setminus \left\{-\frac{1}{h}\right\}$. Also, let $K_{\mathbb{R}}$ be given as in (2.1). We have the following cases.

(i) Suppose $0 < h < \frac{\alpha}{W_0(e^{-1})}$.

(a) If $\lambda \in \left(-\frac{1}{h}, 0\right) \cup (0, \infty)$, then (1.1) is Hyers–Ulam stable, with best HUS constant $K = \frac{1}{|\lambda|}$.

(b) If $\lambda = 0$, then (1.1) is not Hyers–Ulam stable.

(c) If $\lambda \in \left(-\infty, -\frac{1}{h}\right)$, then (1.1) is HUS, with best HUS constant $K = K_{\mathbb{R}}$.

(ii) Suppose $h = \frac{\alpha}{W_0(e^{-1})}$. Then, $(1 + h\lambda)e^{\alpha \lambda} = -1$ at $\lambda = -\frac{1}{\alpha} \left(1 + W_0(e^{-1})\right)$, and we have the following subcases.

(a) If $\lambda \in \left(\frac{W_0(e^{-1})}{-\alpha}, 0\right) \cup (0, \infty)$, then (1.1) is HUS, with best HUS constant $K = \frac{1}{|\lambda|}$.

(b) If $\lambda = 0$ or $\lambda = -\frac{1}{\alpha} \left(1 + W_0(e^{-1})\right)$, then (1.1) is not HUS.

(c) If $\lambda \in \left(-\infty, -\frac{1}{\alpha} \left(1 + W_0(e^{-1})\right)\right) \cup \left(-\frac{1}{\alpha} \left(1 + W_0(e^{-1})\right), \frac{W_0(e^{-1})}{-\alpha}\right)$, then (1.1) is HUS, with best HUS constant $K = K_{\mathbb{R}}$ as in (2.1).
(iii) Suppose $h > \frac{\alpha}{W_0(e^{-1})}$. Then, $(1 + h\lambda)e^{\alpha\lambda} = -1$ at

$$
\lambda_{-1} := -\frac{1}{h} + \frac{1}{\alpha}W_{-1}\left(\frac{-\alpha e^{\pi}}{h}\right) \quad \text{and} \quad \lambda_0 := -\frac{1}{h} + \frac{1}{\alpha}W_0\left(\frac{-\alpha e^{\pi}}{h}\right),
$$

and we have the following subcases.

(a) If $\lambda \in \left(-\frac{1}{h}, 0\right) \cup (0, \infty)$, then (1.1) is HUS, with best HUS constant $K = \frac{1}{|\lambda|}$.

(b) If $\lambda = \lambda_{-1}$, $\lambda = \lambda_0$, or $\lambda = 0$, then (1.1) is not HUS.

(c) If $\lambda \in (-\infty, \lambda_{-1}) \cup (\lambda_{-1}, \lambda_0) \cup (\lambda_0, -\frac{1}{h})$, then (1.1) is HUS, with best HUS constant $K = |K_\mathbb{R}|$ as in (2.1).

Proof. Cases (i)(a), (ii)(a), and (iii)(a) all follow from [2, Corollary 3.8], while cases (i)(b), (ii)(b), and (iii)(b) all follow from [2, Theorem 3.10(ii)].

Case (i)(c). Suppose $\lambda < -\frac{1}{h}$. Since $0 < h < \frac{\alpha}{W_0(e^{-1})}$, the base of the exponential function (1.3) satisfies $(1 + h\lambda)e^{\alpha\lambda} \in (-1, 0)$. Consequently, as $e_\lambda(t, 0) = (1 + h\lambda)^k e^{\alpha\lambda(t-hk)}$, we have an exponential function that changes sign; in particular, $e_\lambda(t, 0) < 0$ for all $t \in [k(\alpha + h), k(\alpha + h) + \alpha]$ when $k$ is odd. Let $\phi$ satisfy the perturbed equation (1.2), and note that

$$
x(t) = \phi_0 e_\lambda(t, 0)
$$

is a well-defined solution of (1.1). Then, for $t \in [k(\alpha + h), k(\alpha + h) + \alpha]$, we have $t = k(\alpha + h) + j$ for some $j \in [0, \alpha]$, and

$$
|\phi(t) - x(t)| = \left| \phi_0 e_\lambda(t, 0) + e_\lambda(t, 0) \int_0^t \frac{q(s)}{e_\lambda(\sigma(s), 0)} \Delta s - \phi_0 e_\lambda(t, 0) \right|
$$

\begin{align*}
&\leq \varepsilon |e_\lambda(t, 0)| \int_0^t \frac{1}{|e_\lambda(\sigma(s), 0)|} \Delta s \\
&= \varepsilon |e_\lambda(t, 0)| \left( \int_0^\alpha + \int_0^{\alpha + h} + \int_0^{2\alpha + h} + \cdots \right. \\
&\quad + \int_0^{k(\alpha + h) - h} + \int_0^{k(\alpha + h)} \frac{1}{|e_\lambda(\sigma(s), 0)|} \Delta s \\
&= \varepsilon |e_\lambda(t, 0)| \left[ \left( \int_0^\alpha + \int_0^{2\alpha + h} + \cdots + \int_0^{k(\alpha + h) - h} \right) \frac{1}{|e_\lambda(\sigma(s), 0)|} ds \\
&\quad + \left( \int_0^{\alpha + h} + \int_0^{2(\alpha + h)} + \cdots + \int_0^{k(\alpha + h)} \right) \frac{\Delta s}{|e_\lambda(\sigma(s), 0)|} \right]
\end{align*}
$$\begin{align*}
&+ \int_{k(\alpha+\varepsilon)+j}^{k(\alpha+\varepsilon)+j} \frac{ds}{|e_\lambda(s,0)|} \\
=\varepsilon |1 + h\lambda|^k e^{\lambda(k\alpha+j)} \left( \sum_{m=0}^{k-1} \frac{e^{\alpha\lambda} - 1}{\lambda |1 + h\lambda|^m} e^{\alpha\lambda} - 1 \right) \\
&+ \sum_{m=1}^{k} \frac{h}{|1 + h\lambda|^m e^{\alpha\lambda m}} \left( \frac{1}{e^{\lambda(k\alpha+j)} - e^{\alpha\lambda}} \right) \\
&\leq \varepsilon \left( \frac{1 + e^{\alpha\lambda} (1 + h\lambda) - 2e^{\lambda}(1 + h\lambda)}{-\lambda (1 + e^{\alpha\lambda}(1 + h\lambda))} \right) \\
&\leq \frac{\varepsilon \left( e^{\alpha\lambda} (1 + h\lambda) - (1 + 2h\lambda) \right)}{-\lambda \left( 1 + e^{\alpha\lambda}(1 + h\lambda) \right)},
\end{align*}$$

where the penultimate line is the result of taking $k \to \infty$, and the last line follows by letting $j = 0$. This shows that (1.1) has HUS with HUS constant at most

$$K_R = \frac{1}{-\lambda} \left( \frac{e^{\alpha\lambda} (1 + h\lambda) - (1 + 2h\lambda)}{1 + e^{\alpha\lambda}(1 + h\lambda)} \right),$$

whenever $\lambda < -\frac{1}{h}$ and we assume $0 < h < \frac{\alpha}{W_0(e^{-1})}$. On the other hand, given any $\varepsilon > 0$, let

$$q(s) := \varepsilon \frac{e_\lambda(\sigma(s),0)}{|e_\lambda(\sigma(s),0)|}, \quad s \in \mathbb{T}.$$  

Clearly $|q(s)| = \varepsilon$ for all $s \in \mathbb{T}$. Using this $q$ in a function $\phi$ of the form (1.5), we have that

$$\phi(t) = \phi_0 e_\lambda(t,0) + \varepsilon e_\lambda(t,0) \int_0^t \frac{\Delta s}{|e_\lambda(\sigma(s),0)|},$$

and $\phi$ satisfies (1.2). Let $x$ the solution of (1.1) with $x_0 = \phi_0$. Let $t = k(\alpha + h)$ ($j = 0$) for arbitrarily large $k \in \mathbb{N}_0$. Then, similar to the calculations done above,

$$\begin{align*}
|\phi(t) - x(t)| &= \varepsilon |e_\lambda(t,0)| \int_0^t \frac{1}{|e_\lambda(\sigma(s),0)|} \Delta s \\
&= \varepsilon |1 + h\lambda|^k e^{\lambda(k\alpha)} \left( \sum_{m=0}^{k-1} \frac{e^{\alpha\lambda} - 1}{\lambda |1 + h\lambda|^m} e^{\alpha\lambda} - 1 \right) \\
&+ \sum_{m=1}^{k} \frac{h}{|1 + h\lambda|^m e^{\alpha\lambda m}} \\
&= \varepsilon \frac{(h\lambda + (-1 + e^{\alpha\lambda})(1 + h\lambda)(-1 + e^{k\alpha\lambda})(1 + h\lambda)^k)}{\lambda(-1 + e^{\alpha\lambda})(1 + h\lambda)},
\end{align*}$$
so that, again with \( t = k(\alpha + h) \), \( k \in \mathbb{N}_0 \) large, and \( \lambda < -\frac{1}{h} \) with \( 0 < h < \frac{\alpha}{W_0(e^{-1})} \), we have

\[
\lim_{t \to \infty} |\phi(t) - x(t)| = \lim_{k \to \infty} \varepsilon \frac{(h\lambda + (-1 + e^{\alpha\lambda})[1 + h\lambda])(-1 + e^{k\alpha\lambda}[1 + h\lambda])}{\lambda(-1 + e^{\alpha\lambda}[1 + h\lambda])}
\]

\[
= \frac{\varepsilon}{-\lambda} \left( \frac{e^{\alpha\lambda}(1 + h\lambda) - (1 + 2h\lambda)}{1 + e^{\alpha\lambda}(1 + h\lambda)} \right),
\]

the same constant as above in (2.1). This proves \( K_\mathbb{R} \) in (2.1) is the best possible constant.

Case (ii)(c). Let \( \lambda \in \left(-\frac{1}{\alpha} \left(1 + W_0(e^{-1})\right), \frac{W_0(e^{-1})}{-\alpha}\right) \). As \( h = \frac{\alpha}{W_0(e^{-1})} \), we have \( (1 + h\lambda)e^{\alpha\lambda} \in (-1, 0) \); to see this, set \( f(\lambda) := (1 + h\lambda)e^{\alpha\lambda} = \left(1 + \frac{\alpha\lambda}{W_0(e^{-1})}\right)e^{\alpha\lambda} \).

Then,

\[
f \left(-\frac{1}{\alpha} \left(1 + W_0(e^{-1})\right)\right) = \left(1 + \frac{-1 - W_0(e^{-1})}{W_0(e^{-1})}\right) e^{-1 - W_0(e^{-1})}
\]

\[
= -1 - W_0(e^{-1}) - e^{-1} = -1
\]

by the property of the Lambert W function; also, \( f \left(\frac{W_0(e^{-1})}{-\alpha}\right) = 0 \). This implies, if \( t = k(\alpha + h) + j \) for \( j \in [0, \alpha] \), then

\[
|e_\lambda(t, 0)| = \left|\left[(1 + h\lambda)e^{\alpha\lambda}\right]^k e^{j\lambda}\right| \leq \left|(1 + h\lambda)e^{\alpha\lambda}\right|^k \leq 1
\]

for any \( k \in \mathbb{N}_0 \), and thus for all \( t \in \mathbb{T} \), and

\[
\lim_{t \to \infty} |e_\lambda(t, 0)| = 0.
\]

The proof of the rest of this case is similar to the proof above of case (i)(c), leading to \( K_\mathbb{R} \) as in (2.1) using the fact that \( \alpha e^{1 + W_0(e^{-1})} = \frac{\alpha}{W_0(e^{-1})} = h \) by the Lambert W function properties. Clearly this is the same \( K \) value as found earlier, in Theorem 2.1 (iii). For \( \lambda \in \left(-\infty, -\frac{1}{\alpha} \left(1 + W_0(e^{-1})\right)\right) \), we also have \( (1 + h\lambda)e^{\alpha\lambda} \in (-1, 0) \). Thus, case (ii)(c) holds.

Case (iii)(c). Let the exponential function be given by (1.3). For \( \lambda \in (\lambda_-, \lambda_0) \), the base of the exponential function satisfies \( (1 + h\lambda)e^{\alpha\lambda} < -1 \). If \( \phi \) satisfies (1.2), then \( \phi \) has the form given in (1.5), and

\[
\int_0^\infty \frac{q(s)}{e_\lambda(\sigma(s), 0)} \Delta s
\]
exists and is finite. Note that
\[
x(t) = x_0 e^{\lambda(t, 0)} \text{ for } x_0 = \phi_0 + \int_0^\infty \frac{q(s)}{e^{\lambda(s, 0)}} \Delta s
\]
is a well-defined solution of (1.1), and
\[
|\phi(t) - x(t)| = |e^{\lambda(t, 0)}| \left| - \int_t^\infty \frac{q(s)}{e^{\lambda(s, 0)}} \Delta s \right|
\leq \varepsilon |e^{\lambda(t, 0)}| \int_t^\infty \frac{1}{|e^{\lambda(s, 0)}|} \Delta s
= \varepsilon \left| 1 + h\lambda \right|^k e^{\lambda(ka+j)} \left( \sum_{m=k+1}^{\infty} \frac{e^{\alpha\lambda} - 1}{\lambda[1 + h\lambda]^{m} e^{\alpha\lambda(m+1)}} \right)
+ \sum_{m=k+1}^{\infty} \frac{h}{[1 + h\lambda]^{m} e^{\alpha\lambda m}} - \frac{1}{\lambda[1 + h\lambda]^{k}} \left( \frac{1}{e^{\lambda(ka+j)}} - \frac{1}{e^{\lambda(ka+j)}} \right)
= \frac{\varepsilon \left( -1 + e^{\lambda(1 + h\lambda)} + (e^{\alpha\lambda} - e^{j\lambda})[1 + h\lambda] \right)}{\lambda (-1 + e^{\alpha\lambda}[1 + h\lambda])}
\leq \frac{\varepsilon \left( (1 + 2h\lambda) - e^{\alpha\lambda}(1 + h\lambda) \right)}{-\lambda(1 + e^{\alpha\lambda}(1 + h\lambda))}
= |K| \varepsilon
\]
for \( K = K_R \) in (2.1), having taken \( j = 0 \) to get the penultimate line. □

**Example 2.2.** Let \( \lambda = -\frac{2}{h} \) for \( 0 < h < \frac{\alpha}{W_0(e^{-1})} \). By Theorem 2.1(i)(c), (1.1) is Hyers–Ulam stable, with minimal HUS constant
\[
K = \frac{h(3e^{\frac{2}{h}} - 1)}{2(e^{\frac{2}{h}} - 1)} = \frac{h}{2} \left( 2 + \coth \left( \frac{\alpha}{h} \right) \right),
\]
where we have used \( K = K_R \) in (2.1). ♦

**Remark 2.3.** Fix the jump size \( h > \frac{\alpha}{W_0(e^{-1})} \), as above in Theorem 2.1(iii), and let \( \alpha \) tend to 0. Then,
\[
\lim_{\alpha \to 0} P_{\alpha,h} = h\mathbb{Z}.
\]
Note that, for \( \alpha = 0 \), \( (1 + h\lambda) = -1 \) at \( \lambda_0 := \lim_{\alpha \to 0} -\frac{1}{h} + \frac{1}{\alpha} W_0 \left( -\frac{\alpha}{h} e^{rac{2}{h}} \right) = -\frac{2}{h} \), \( \lim_{\alpha \to 0} \lambda_{-1} = -\infty \), and we have the following subcases from Theorem 2.1(iii), with \( \alpha = 0 \).

(a) If \( \lambda \in \left( -\frac{1}{h}, 0 \right) \cup (0, \infty) \), then (1.1) is HUS, with best HUS constant \( K = \frac{1}{|\lambda|} \).
(b) If $\lambda = -\frac{2}{h}$, and $\lambda = 0$, then (1.1) is not HUS.

(c) If $\lambda \in \left(-\infty, -\frac{2}{h}\right) \cup \left(-\frac{2}{h}, -\frac{1}{h}\right)$, then (1.1) is HUS, with best HUS constant $K = \frac{1}{|\lambda + \frac{2}{h}|}$.

Case (iii)(c) recovers the $K$ value found in the $h\mathbb{Z}$ case in Onitsuka [23, Remark 4.6]; see also [4, Theorem 2.6].

Remark 2.4. If one does the analogous analysis on $P_{\alpha,h}$ using the nabla backward difference operator instead of the Delta forward difference operator, then similarly interesting results are obtained. In the nabla case, the nabla differential operator is defined by

$$x^{\nabla}(t) = \begin{cases} \frac{d}{dt}x(t) : t \in (k(\alpha + h), k(\alpha + h) + \alpha] \\ \frac{x(t) - x(t-h)}{h} : t = k(\alpha + h), \end{cases}$$

and the nabla exponential function is given via

$$\hat{e}_\lambda(t,0) = \frac{e^{\lambda t}}{(1-h\lambda)e^{h\lambda}}k, \quad \lambda \in \mathbb{C} \setminus \left\{ \frac{1}{h} \right\}.$$

If we take

$$\hat{K}_R = \frac{h\lambda - 1 + e^{\alpha\lambda}(1-2h\lambda)}{\lambda(h\lambda - 1 - e^{\alpha\lambda})}.$$

for the nabla dynamic equation

$$x^{\nabla}(t) = \lambda x(t), \quad \lambda \in \mathbb{C} \setminus \left\{ \frac{1}{h} \right\}, \quad t \in \mathbb{T},$$

compare the following theorem with Theorem 2.1.

Theorem 2.5 (Nabla equation). Fix $\alpha > 0$, and let $\lambda \in \mathbb{R} \setminus \{1/h\}$. Also, let $\hat{K}_R$ be given as in (2.3). We have the following cases.

(i) Suppose $0 < h < \frac{\alpha}{W_0(e^{-1})}$.

(a) If $\lambda \in (-\infty, 0) \cup \left(0, \frac{1}{h}\right)$, then (2.4) is Hyers–Ulam stable, with best HUS constant $K = \frac{1}{|\lambda|}$.

(b) If $\lambda = 0$, then (2.4) is not Hyers–Ulam stable.
(c) If $\lambda \in \left(\frac{1}{h}, \infty\right)$, then (2.4) is HUS, with best HUS constant $K = \tilde{K}_R$.

(ii) Suppose $h = \frac{\alpha}{W_0(e^{-1})}$. Then, $(1 - h\lambda)^{-1}e^{\alpha\lambda} = -1$ at $\lambda = \frac{1}{\alpha} \left(1 + W_0(e^{-1})\right)$, and we have the following subcases.

(a) If $\lambda \in (-\infty, 0) \cup \left(0, \frac{1}{h}\right)$, then (2.4) is HUS, with best HUS constant $K = \frac{1}{|\lambda|}$.

(b) If $\lambda = 0$ or $\lambda = \frac{1}{\alpha} \left(1 + W_0(e^{-1})\right)$, then (2.4) is not HUS.

(c) If $\lambda \in \left(\frac{W_0(e^{-1})}{\alpha}, \frac{1}{\alpha} \left(1 + W_0(e^{-1})\right)\right) \cup \left(\frac{1}{\alpha} \left(1 + W_0(e^{-1})\right), \infty\right)$, then (2.4) is HUS, with best HUS constant $K = \tilde{K}_R$ as in (2.3).

(iii) Suppose $h > \frac{\alpha}{W_0(e^{-1})}$. Then, $(1 - h\lambda)^{-1}e^{\alpha\lambda} = -1$ at

$$\hat{\lambda}_{-1} := \frac{1}{h} - \frac{1}{\alpha} W_{-1}\left(-\frac{\alpha}{h} e^\pi\right) \quad \text{and} \quad \hat{\lambda}_0 := \frac{1}{h} - \frac{1}{\alpha} W_0\left(-\frac{\alpha}{h} e^\pi\right),$$

and we have the following subcases.

(a) If $\lambda \in (-\infty, 0) \cup \left(0, \frac{1}{h}\right)$, then (2.4) is HUS, with best HUS constant $K = \frac{1}{|\lambda|}$.

(b) If $\lambda = 0$, $\lambda = \hat{\lambda}_0$, or $\lambda = \hat{\lambda}_{-1}$, then (2.4) is not HUS.

(c) If $\lambda \in \left(\frac{1}{h}, \hat{\lambda}_0\right) \cup \left(\hat{\lambda}_0, \hat{\lambda}_{-1}\right) \cup \left(\hat{\lambda}_{-1}, \infty\right)$, then (2.4) is HUS, with best HUS constant $K = |\tilde{K}_R|$ as in (2.3).

Remark 2.6. If we compare $K_R$ in (2.1) with $\tilde{K}_R$ in (2.3), we see that

$$|K_R(\lambda)| = \left|\tilde{K}_R(-\lambda)\right|,$$

where we have made them into functions of the parameter $\lambda$. ◊
3 Complex Eigenvalues

In this section, we extend the considered values of the eigenvalue to \( \lambda \in \mathbb{C} \setminus \{-\frac{1}{h}\} \) on the time scale \( \mathbb{T}_{\alpha,h} \) for continuous interval size \( \alpha > 0 \) and discrete jump size \( h > 0 \). To further motivate our use of the Lambert \( W \) function, consider the exponential function given in (1.3). Set the base of the exponential function as follows, \((1 + h\lambda)e^{\alpha\lambda} = \text{Re}i\theta\), for \( R > 0 \), \( i = \sqrt{-1} \), and \( \theta \in (-\pi, \pi] \). Let \( w = \frac{\alpha}{h} + \alpha \lambda \). Then, the following are equivalent:

\[
(1 + h\lambda)e^{\alpha\lambda} = \text{Re}i\theta \\
\left(\frac{\alpha}{h} + \alpha \lambda\right)e^{\alpha\lambda} = \frac{R\alpha}{h}e^{i\theta} \\
w e^w = \frac{R\alpha}{h}e^{\frac{\alpha}{h}+i\theta} \\
w = W_z\left(\frac{R\alpha}{h}e^{\frac{\alpha}{h}+i\theta}\right),
\]

so that

\[
\lambda = -\frac{1}{h} + \frac{1}{\alpha}W_z\left(\frac{R\alpha}{h}e^{\frac{\alpha}{h}+i\theta}\right), \quad \theta \in (-\pi, \pi], \quad R > 0, \quad h > 0, \quad (3.1)
\]

for various branches of the Lambert \( W \) function in the complex plane determined by \( z \in \mathbb{Z} \), for \( \theta \in (-\pi, \pi] \), with a branch cut along the negative real axis, and principal branch \( W_0 \).

**Theorem 3.1** (Delta equation). Let \( \lambda \in \mathbb{C} \setminus \{-\frac{1}{h}\} \) have the form (3.1), and let \( W_z \) be the Lambert \( W \) function for any \( z \in \mathbb{Z} \).

(i) If \( R = 1 \), then (1.1) is not Hyers–Ulam stable.

(ii) If \( R > 1 \), then (1.1) is Hyers–Ulam stable, with HUS constant at most

\[
K_C := \max_{j \in [0, \alpha]} \frac{R - 1 - Re^{(j-\alpha)Re(\lambda)} + e^{jRe(\lambda)} (1 + h Re(\lambda))}{(R - 1) Re(\lambda)}, \quad (3.2)
\]

or \( K_C = \frac{h + R\alpha}{R - 1} \) if \( \text{Re}(\lambda) = 0 \).

(iii) If \( 0 < R < 1 \), then (1.1) is Hyers–Ulam stable, with HUS constant at most

\[
|K_C| = \max_{j \in [0, \alpha]} \frac{R - 1 - Re^{(j-\alpha)Re(\lambda)} + e^{jRe(\lambda)} (1 + h Re(\lambda))}{(1 - R) Re(\lambda)}, \quad (3.3)
\]
Proof. Let $\lambda \in \mathbb{C} \setminus \left\{ -\frac{1}{h} \right\}$ have the form (3.1), and let $W_z$ be the Lambert $W$ function for any $z \in \mathbb{Z}$.

Case (i). If $R = 1$, then $\lambda = -\frac{1}{h} + \frac{1}{\alpha} W_z \left( \frac{\alpha}{h} e^{\frac{\alpha}{h} + i\theta} \right)$ for $\theta \in (-\pi, \pi]$ and for fixed $z \in \mathbb{Z}$. Let the exponential function be given by (1.3). Then, for $t = k(\alpha + h) + j \in [k(\alpha + h), k(\alpha + h) + \alpha]$ and $j \in [0, \alpha]$,

$$e_\lambda(t, 0) = \left[(1 + h\lambda)e^{\alpha t}\right]^{k} e^{j\lambda} = e^{j\lambda + ik\theta}.$$ 

Note that, for all $j \in [0, \alpha]$ and $\theta \in (-\pi, \pi]$, and for any fixed $z \in \mathbb{Z}$, the real part of $\lambda$ satisfies $\text{Re}(\lambda) \leq 0$, and

$$|e_\lambda(t, 0)| = e^{j\text{Re}(\lambda)} \in [e^{\alpha\text{Re}(\lambda)}, 1].$$

So, with $e_\lambda(t, 0) = e^{j\lambda + ik\theta}$ for $t = k(\alpha + h) + j$, $j \in [0, \alpha]$, and $\theta \in (-\pi, \pi]$, set $\phi(t) = \epsilon e_\lambda(t, 0)$. Then, we have

$$|\phi^\Delta(t) - \lambda \phi(t)| = |\epsilon e_\lambda e_\lambda(t, 0) + \epsilon e_\lambda^\Delta(t, 0) - \epsilon \lambda e_\lambda(t, 0)| = \epsilon |e_\lambda^\alpha(t, 0)| \leq \epsilon$$

implies that $\phi$ satisfies (1.2), so that

$$|\phi(t) - x(t)| = |e_\lambda(t, 0)||\epsilon t - x_0| \geq e^{\alpha\text{Re}(\lambda)}|\epsilon t - x_0| \to \infty$$

for any possible initial condition $x_0$, meaning (1.1) is not HUS for $R = 1$, that is when

$$\lambda = -\frac{1}{h} + \frac{1}{\alpha} W_z \left( \frac{\alpha}{h} e^{\frac{\alpha}{h} + i\theta} \right)$$

for any $\theta \in (-\pi, \pi]$, $h > 0$, and for any fixed $z \in \mathbb{Z}$.

Case (ii). Let $R > 1$, that is, let $\lambda = -\frac{1}{h} + \frac{1}{\alpha} W_z \left( \frac{R\alpha}{h} e^{\frac{R\alpha}{h} + i\theta} \right)$, initially with $\text{Re}(\lambda) \neq 0$, for $\theta \in (-\pi, \pi]$ and $z \in \mathbb{Z}$. Let the exponential function be given by (1.3), and let $\phi$ satisfy (1.2). Then, $\phi$ has the form given in (1.5), and again,

$$\int_0^\infty \frac{q(s)}{e_\lambda(\sigma(s), 0)} \Delta s$$

exists and is finite, as $|e_\lambda(t, 0)| = R^k e^{j\text{Re}(\lambda)}$ for $R > 1$ and $t = k(\alpha + h) + j$, $j \in [0, \alpha]$. Note that

$$x(t) = x_0 e_\lambda(t, 0), \quad x_0 = \phi_0 + \int_0^\infty \frac{q(s)}{e_\lambda(\sigma(s), 0)} \Delta s$$

is a well-defined solution of (1.1). Now, to integrate from $s = 0$ to $s = t = k(\alpha + h) + j$ for some $k \in \{0, 1, 2, \ldots\}$ and $j \in [0, \alpha]$, we see that there are $k$ continuous intervals and $k$ gaps to integrate over, plus the final partial interval (continuous), so that

$$\int_0^t \frac{\Delta s}{|e_\lambda(\sigma(s), 0)|} = \sum_{m=0}^{k-1} \left( \frac{1}{|e_\lambda(s, 0)|} \int_{m(\alpha + h)}^{(m+1)(\alpha + h) + \alpha} ds \right).$$
Using these two integral values, we have

\[
\begin{align*}
&= \sum_{m=1}^{k} \left( \int_{m(h+\alpha)}^{m(h+\alpha)-\Delta} \frac{ds}{\epsilon_\lambda(s,0)} \right) + \int_{k(h+\alpha)}^{t} \frac{ds}{\epsilon_\lambda(s,0)} \\
&= \sum_{m=0}^{k-1} \left( \int_{mR^2}^{\alpha} \frac{e^{-j Re(\lambda)}}{R^m} dj \right) + \sum_{m=1}^{k} \frac{h}{R^m} + \int_{0}^{j} \frac{e^{-\ell Re(\lambda)}}{R^k} d\ell \\
&= \sum_{m=0}^{k-1} \frac{-1}{R^m Re(\lambda)} e^{-j Re(\lambda)} \left| \int_{j=0}^{\alpha} + \sum_{m=1}^{k} \frac{h}{R^m} + \frac{-1}{R^k Re(\lambda)} e^{-j Re(\lambda)} \right| j = 0 \\
&= \frac{R(R^k - 1)(1 - e^{-\alpha Re(\lambda)})}{R^k(R - 1)Re(\lambda)} + \frac{h(R^k - 1)}{R^k(R - 1)} + \frac{1 - e^{-j Re(\lambda)}}{R^k Re(\lambda)} (3.4)
\end{align*}
\]

and

\[
\int_{0}^{\infty} \frac{\Delta s}{|\epsilon_\lambda(s,0)|} = \lim_{t \to \infty} \int_{0}^{t} \frac{\Delta s}{|\epsilon_\lambda(s,0)|} = \frac{R(1 - e^{-\alpha Re(\lambda)})}{(R - 1)Re(\lambda)} + \frac{h}{R - 1}.
\]

Using these two integral values, we have

\[
|\phi(t) - x(t)| = |\epsilon_\lambda(t,0)| - \int_{t}^{\infty} q(s) \frac{\Delta s}{\epsilon_\lambda(s,0)} \Delta s \\
\leq \varepsilon |\epsilon_\lambda(t,0)| \int_{t}^{\infty} \frac{1}{|\epsilon_\lambda(s,0)|} \Delta s \\
= \varepsilon |\epsilon_\lambda(t,0)| \left( \int_{0}^{\infty} - \int_{0}^{t} \right) \frac{1}{|\epsilon_\lambda(s,0)|} \Delta s \\
= \varepsilon \left( \frac{R - 1 + e^{j Re(\lambda)} - Re^{(j - \alpha) Re(\lambda)} + e^{j Re(\lambda)} h Re(\lambda)}{(R - 1)Re(\lambda)} \right)
\]

for \( j \in [0, \alpha] \), and for fixed \( z \in \mathbb{Z}, R > 1, \theta \in (-\pi, \pi) \) that determine \( \lambda \in \mathbb{C} \) with \( Re(\lambda) \neq 0 \). Set \( K \) as in (3.2), that is,

\[
K_C := \max_{j \in [0, \alpha]} \frac{R - 1 + e^{j Re(\lambda)} - Re^{(j - \alpha) Re(\lambda)} + e^{j Re(\lambda)} h Re(\lambda)}{(R - 1)Re(\lambda)}.
\]

Therefore, (1.1) has HUS for \( \lambda = -\frac{1}{h} + \frac{1}{\alpha} W_z \left( \frac{R\alpha}{h} e^{\pi + \theta} \right) \) with \( Re(\lambda) \neq 0 \) and \( R > 1 \), with HUS constant at most \( K_C \). If \( Re(\lambda) = 0 \), then

\[
\lim_{Re(\lambda) \to 0} K_C = \lim_{Re(\lambda) \to 0} \max_{j \in [0, \alpha]} \frac{R - 1 + e^{j Re(\lambda)} - Re^{(j - \alpha) Re(\lambda)} + e^{j Re(\lambda)} h Re(\lambda)}{(R - 1)Re(\lambda)}
\]

\[
= \max_{j \in [0, \alpha]} \frac{h + Ra + j(1 - R)}{R - 1}
\]

\[
= \frac{h + Ra}{R - 1}.
\]
Re(3.4), as well as $t = 3.2$

Remark 14

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to $\alpha$ for $R$ solution of (1.1) with form (1.4), where

$$\lambda > 0$$ in (3.2). This ends the proof.

$h R + \int_0^t \Delta s |e_\lambda(\sigma(s), 0)| = \left| R(1 - R^k)(1 - e^{-\alpha \text{Re}(\lambda)}) R^k(1 - R) \text{Re}(\lambda) + \frac{h(1 - R^k)}{R^k(1 - R)} + \frac{1 - e^{-j \text{Re}(\lambda)}}{R^k \text{Re}(\lambda)} \right| (3.5)$

If $\phi$ satisfies the perturbed equation (1.2), then $\phi$ is again given as in (1.5). Let $x$ be a solution of (1.1) with form (1.4), where

$$x_0 = \phi_0 - \varepsilon \left( \frac{h}{1 - R} + \frac{R(1 - e^{-\alpha \text{Re}(\lambda)})}{(1 - R) \text{Re}(\lambda)} \right)$$

note that both fractions in the parentheses here are positive, due to $R \in (0, 1)$ and $\text{Re}(\lambda) < 0$ in this case. Employing (3.5) with $t = k(\alpha + h) + j$, we see that

$$|\phi(t) - x(t)| = |e_\lambda(t, 0)| \left| \phi_0 + \int_0^t \frac{q(s)}{e_\lambda(\sigma(s), 0)} \Delta s \right.$$ 

$$- \left( \phi_0 - \varepsilon \left( \frac{h}{1 - R} + \frac{R(1 - e^{-\alpha \text{Re}(\lambda)})}{(1 - R) \text{Re}(\lambda)} \right) \right) \right|$$ 

$$= |e_\lambda(t, 0)| \left| \int_0^t \frac{q(s)}{e_\lambda(\sigma(s), 0)} \Delta s + \varepsilon \left( \frac{h}{1 - R} + \frac{R(1 - e^{-\alpha \text{Re}(\lambda)})}{(1 - R) \text{Re}(\lambda)} \right) \right|$$ 

$$\leq \varepsilon |e_\lambda(t, 0)| \left| \int_0^t \frac{1}{e_\lambda(\sigma(s), 0)} \Delta s + \frac{h}{1 - R} + \frac{R(1 - e^{-\alpha \text{Re}(\lambda)})}{(1 - R) \text{Re}(\lambda)} \right|$$ 

$$= \varepsilon R^k e^{j \text{Re}(\lambda)} \left( \frac{R(1 - e^{-\alpha \text{Re}(\lambda)})}{R^k(1 - R) \text{Re}(\lambda)} + \frac{h}{R^k(1 - R)} + \frac{1 - e^{-j \text{Re}(\lambda)}}{R^k \text{Re}(\lambda)} \right)$$ 

$$= \varepsilon e^{j \text{Re}(\lambda)} \left( \frac{R(1 - e^{-\alpha \text{Re}(\lambda)})}{(1 - R) \text{Re}(\lambda)} + \frac{h}{1 - R} + \frac{1 - e^{-j \text{Re}(\lambda)}}{\text{Re}(\lambda)} \right)$$ 

$$\leq \varepsilon \left( \frac{1 - R + R e^{(j - \alpha) \text{Re}(\lambda)} - e^{j \text{Re}(\lambda)}(1 + h \text{Re}(\lambda))}{(-1 + R) \text{Re}(\lambda)} \right)$$

for $j \in [0, \alpha]$, as $R \in (0, 1)$. Therefore (1.1) has HUS for $\lambda = -\frac{1}{h} + \frac{1}{\alpha} W z \left( \frac{R\alpha}{h} e^{\frac{\alpha}{2} + i\theta} \right)$ for $R \in (0, 1)$, with HUS constant given by at most $K = |K_\mathcal{C}|$ given in (3.3), for $K_\mathcal{C}$ as in (3.2). This ends the proof.

Remark 3.2. In Figure 3.1, we illustrate the effects of an increasing jump size $h$, relative to $\alpha$, on the eigenvalues $\lambda$ as parametrized curves in the complex plane. Here, $\alpha = 1$,
Figure 3.1: Delta case: Let $\alpha = 1$, and let $\lambda$ be as in (3.1), for all $\theta \in (-\pi, \pi]$ and $z = -1$ (orange), $z = 0$ (cyan), $z = 1$ (blue). Let $R = \frac{1}{2}$ (left-hand curve and oval), $R = 1$ (red middle curve, unstable manifold), and $R = 2$ (right-hand curve). **Left Graph:** $h = 3.0$ The parametrized values of $\lambda \in \mathbb{C}$ before the bifurcation in the unstable manifold has occurred. **Middle Graph:** $h = \frac{1}{W_0(e^{-1})} \approx 3.59112$ (the bifurcation value) The unstable manifold is the homoclinic orbit given by the parametrized graph of $\lambda = -W_0(e^{-1}) + W_2(e^{1+i\theta})$. **Right Graph:** $h = 3.7$ The parametrized values of $\lambda \in \mathbb{C}$ after the bifurcation in the unstable manifold has occurred. End of caption.

and $h$ increases from $h = 3.0$, through the bifurcation value of $h = \frac{1}{W_0(e^{-1})}$, to $h = 3.7$, after the bifurcation in the parameter space has occurred. In Figure 3.2, the complex eigenvalues for the nabla equation are likewise illustrated.

**Remark 3.3.** For $\lambda$ as given in (3.1), note that $R = |1 + h\lambda| e^{\alpha \text{Re}(\lambda)}$. As the jump size $h > 0$ approaches zero with $\alpha > 0$ fixed, $R = e^{\alpha \text{Re}(\lambda)}$ implies $\text{Re}(\lambda) = \frac{1}{\alpha} \ln R$. If we write $\lambda = \text{Re}(\lambda) + \text{Im}(\lambda)i$, where $\text{Re}$ and $\text{Im}$ are the real and imaginary parts of $\lambda \in \mathbb{C} \setminus \left\{-\frac{1}{h}\right\}$, respectively, then

$$R \cos(\theta) = e^{\alpha \text{Re}(\lambda)} \left[(1 + h \text{Re}(\lambda)) \cos(\text{Im}(\lambda)) - h \text{Im}(\lambda) \sin(\text{Im}(\lambda))\right],$$
\( R \sin(\theta) = e^{\alpha \text{Re}(\lambda)} [(1 + h \text{Re}(\lambda)) \sin(\text{Im}(\lambda)) + h \text{Im}(\lambda) \cos(\text{Im}(\lambda))] \);

taking \( h \) to zero, we see that \( \sin(\theta) = \sin(\text{Im}(\lambda)) \) and \( \cos(\theta) = \cos(\text{Im}(\lambda)) \). Summarizing, we have that for fixed \( \alpha > 0 \),

\[
\lim_{h \to 0^+} \mathbb{P}_{\alpha,h} = \mathbb{R}
\]

and

\[
\lim_{h \to 0^+} \lambda = \frac{1}{\alpha} \ln R + (\theta + 2\pi z)i, \quad R > 0, \quad \theta \in (-\pi, \pi], \quad z \in \mathbb{Z}.
\]

In particular, note that for \( R = 1 \), the eigenvalues \( \lambda \) in (3.1) converge to purely imaginary points in the complex plane as the jump size \( h \) goes to zero, which corresponds to the known fact that the Hyers–Ulam instability region for (1.1) with \( T = \mathbb{R} \) is the imaginary axis. See Theorem 3.1(i).

**Remark 3.4.** Similar to Remark 2.3 earlier, fix the jump size \( h > \alpha W_0(e^{-1}) \), and let \( \alpha \) tend to 0. Then, \( \lim_{\alpha \to 0} \mathbb{P}_{\alpha,h} = h\mathbb{Z} \), and we have the following cases from Theorem 3.1, with \( \alpha = 0 \), along the principal branch of the Lambert W function, \( W_0 \).

(i) If \( R = 1 \), then (1.1) is not Hyers–Ulam stable.

(ii) If \( R > 1 \), then (1.1) is Hyers–Ulam stable, with HUS constant at most

\[
K_C = \frac{h}{R - 1} = \frac{h}{|1 + h\lambda| - 1}.
\]

(iii) If \( 0 < R < 1 \), then (1.1) is Hyers–Ulam stable, with HUS constant at most

\[
K = |K_C| = \frac{h}{1 - R} = \frac{h}{1 - |1 + h\lambda|}.
\]

Thus, we recover the (best) \( K \) value found in the \( h\mathbb{Z} \) case in [4, Theorem 2.6]. Since

\[
|K_C| = \frac{h}{|1 + h\lambda| - 1} = \frac{1}{|\text{Re}_h(\lambda)|},
\]

which is the absolute value of the reciprocal of the Hilger-real part of \( \lambda \), we can view \( K_C^{-1} \) as the \( \mathbb{P}_{\alpha,h} \)-real part of \( \lambda \) in some sense.

**Remark 3.5.** Consider the nabla case with \( \lambda \in \mathbb{C}\setminus\{1/h\} \) on the time scale \( \mathbb{P}_{\alpha,h} \), for continuous interval size \( \alpha > 0 \) and discrete jump size \( h > 0 \). With the nabla exponential function given in (2.2), set the base of the exponential function as follows, \((1 - h\lambda)^{-1} e^{\alpha \lambda} = R e^{i\theta}\), for \( R > 0 \), \( i = \sqrt{-1} \), and \( \theta \in (-\pi, \pi] \). Let \( w = \frac{\alpha}{h} - \alpha \lambda \). Then, the following are equivalent:

\[
(1 - h\lambda)^{-1} e^{\alpha \lambda} = R e^{i\theta}
\]

\[
\left(\frac{\alpha}{h} - \alpha \lambda\right) e^{-\alpha \lambda} = \frac{\alpha}{h R} e^{-i\theta}
\]

\[
\frac{(\alpha - h\lambda)^{-1} e^{\alpha \lambda}}{\alpha - h\lambda} = \frac{1}{h R} e^{-i\theta}
\]

\[
\frac{(\alpha - h\lambda)^{-1} e^{\alpha \lambda}}{\alpha - h\lambda} = \frac{1}{h R} e^{-i\theta}
\]

\[
\frac{(\alpha - h\lambda)^{-1} e^{\alpha \lambda}}{\alpha - h\lambda} = \frac{1}{h R} e^{-i\theta}
\]

\[
\frac{(\alpha - h\lambda)^{-1} e^{\alpha \lambda}}{\alpha - h\lambda} = \frac{1}{h R} e^{-i\theta}
\]

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\frac{(\alpha - h\lambda)^{-1} e^{\alpha \lambda}}{\alpha - h\lambda} = \frac{1}{h R} e^{-i\theta}
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\frac{(\alpha - h\lambda)^{-1} e^{\alpha \lambda}}{\alpha - h\lambda} = \frac{1}{h R} e^{-i\theta}
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\frac{(\alpha - h\lambda)^{-1} e^{\alpha \lambda}}{\alpha - h\lambda} = \frac{1}{h R} e^{-i\theta}
\]

\[
\frac{(\alpha - h\lambda)^{-1} e^{\alpha \lambda}}{\alpha - h\lambda} = \frac{1}{h R} e^{-i\theta}
\]

\[
\frac{(\alpha - h\lambda)^{-1} e^{\alpha \lambda}}{\alpha - h\lambda} = \frac{1}{h R} e^{-i\theta}
\]
Figure 3.2: Nabla case: Let $\alpha = 1$, and let $\lambda$ be as in (3.6), for all $\theta \in (-\pi, \pi]$ and $z = -1$ (black), $z = 0$ (cyan), $z = 1$ (brown). Let $R = \frac{1}{2}$ (left-hand curve), $R = 1$ (orange/red/pink middle curves, unstable manifold), and $R = 2$ (oval and right-hand curve). **Left Graph:** $h = 3.0$ The parametrized values of $\lambda \in \mathbb{C}$ before the bifurcation in the unstable manifold has occurred. **Middle Graph:** $h = \frac{1}{W_0(e^{-1})} \approx 3.59112$ (the bifurcation value) The unstable manifold is the homoclinic orbit given by the parametrized graph of $\lambda = -W_0(e^{-1}) + W_z(e^{1+i\theta})$. **Right Graph:** $h = 3.7$ The parametrized values of $\lambda \in \mathbb{C}$ after the bifurcation in the unstable manifold has occurred. End of caption.

\[
we^w = \frac{\alpha}{hR} e^{\frac{\alpha}{hR}e^{-i\theta}}
\]

so that

\[
\lambda = \frac{1}{h} - \frac{1}{\alpha} W_z \left( \frac{\alpha}{hR} e^{\frac{\alpha}{hR}e^{-i\theta}} \right), \quad \theta \in (-\pi, \pi], \quad R > 0, \quad h > 0,
\]

for various branches of the Lambert $W$ function in the complex plane determined by $z \in \mathbb{Z}$, for $\theta \in (-\pi, \pi]$, with a branch cut along the negative real axis, and principal branch $W_0$. Compare the following theorem with Theorem 3.1.
Theorem 3.6 (Nabla equation). Let $\lambda \in \mathbb{C}\setminus\{1/h\}$ have the form (3.6), and let $W_z$ be the Lambert $W$ function for any $z \in \mathbb{Z}$.

(i) If $R = 1$, then (2.4) is not Hyers–Ulam stable.

(ii) If $R > 1$, then (2.4) is Hyers–Ulam stable, with HUS constant at most

$$K = \left| \hat{K}_{\mathbb{C}} \right| := \max_{j \in [0,\alpha]} \frac{R - 1 - R e^{(j - \alpha) \Re(\lambda)} + e^{j \Re(\lambda)} (1 + R h \Re(\lambda))}{(R - 1) \Re(\lambda)},$$

or $\hat{K}_{\mathbb{C}} = \frac{R(h + \alpha)}{R - 1}$ if $\Re(\lambda) = 0$.

(iii) If $0 < R < 1$, then (2.4) is Hyers–Ulam stable, with HUS constant at most

$$K = \left| \hat{K}_{\mathbb{C}} \right| := \max_{j \in [0,\alpha]} \frac{R - 1 - R e^{(j - \alpha) \Re(\lambda)} + e^{j \Re(\lambda)} (1 + R h \Re(\lambda))}{(1 - R) \Re(\lambda)}.$$

4 Related Time Scales

Related to the time scale $T = \mathbb{P}_{\alpha,h}$ are time scales with continuous intervals broken up by isolated points. For example, consider the time scale

$$T = \mathbb{P}_{\alpha,\beta,\gamma,\delta} := \bigcup_{k=0}^{\infty} [k(\alpha,\beta,\gamma,\delta), k(\alpha,\beta,\gamma,\delta) + \alpha]$$

$$\cup \{k(\alpha,\beta,\gamma,\delta) + (\alpha + \beta)\} \cup \{k(\alpha,\beta,\gamma,\delta) + (\alpha + \beta + \gamma)\},$$

which one can think of as dash-dot-dot, dash-dot-dot, and so on, a continuous dash or interval of length $\alpha$, followed by jumps of length $\beta, \gamma$ to two isolated points, respectively, followed by a jump of length $\delta$ to the next continuous interval, repeated.

Theorem 4.1. Let $\mathcal{I}_k = [k(\alpha,\beta,\gamma,\delta), k(\alpha,\beta,\gamma,\delta) + \alpha]$. The solution to

$$x^{\Delta}(t) = \lambda x(t), \quad t \in \mathbb{P}_{\alpha,\beta,\gamma,\delta},$$

is given by the exponential function

$$e_{\lambda}(t, 0) = \begin{cases} \left( \frac{(1 + \beta \lambda)(1 + \gamma \lambda)(1 + \delta \lambda)}{e^{(\beta + \gamma + \delta)\lambda}} \right)^{k} e^{\lambda t} & \text{if } t \in \mathcal{I}_k \\ (1 + \beta \lambda)^{k+1} \left( (1 + \gamma \lambda)(1 + \delta \lambda) \right)^{k} e^{\alpha \lambda(k+1)} & \text{if } t = T_{k,\alpha,\beta} \\ ((1 + \beta \lambda)(1 + \gamma \lambda))^{k+1} (1 + \delta \lambda)^{k} e^{\alpha \lambda(k+1)} & \text{if } t = T_{k,\alpha,\beta,\gamma} \end{cases}$$

for each fixed $k \in \mathbb{N}_0$ and $t \in \mathbb{P}_{\alpha,\beta,\gamma,\delta}$, where $T_{k,\alpha,\beta} = k(\alpha,\beta,\gamma,\delta) + (\alpha + \beta)$ and

$$T_{k,\alpha,\beta,\gamma} = k(\alpha,\beta,\gamma,\delta) + (\alpha + \beta + \gamma).$$

The HUS analysis for this time scale would clearly track with the analysis earlier in this work, and involve the Lambert $W$ function.
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References


