Practice Exam 4
Modern Algebra
Wednesday, November 24, 2010

Show all your work and justify all your answers. Be sure to write your proofs in complete sentences, whether in English words or symbolic notation. If you define a function or operation based on cosets, be sure to show that it is well-defined. If you define a homomorphism, be sure to show that it is homomorphic. Make your examples concrete and specific.

1. Give a specific example of a group $G$ and a subgroup $H$ of $G$ such that $H$ is not normal in $G$. Be sure to demonstrate that $H \not\triangleleft G$. [10 points]

2. Let $G$ be a group, and let both $N$ and $K$ be normal subgroups of $G$. Let $NK = \{nk : n \in N, k \in K\}$.

It is a fact that $NK$ is a subgroup of $G$. Prove that $NK$ is normal in $G$. [10 points]

3. I have a supply of black and white beads and I want to make some 9-bead necklaces. The beads will be simply strung on a circular piece of string, as shown in the picture below. Of course, two necklaces would be the same if I could rotate or flip one to look like the other. How many different necklaces can I make? [10 points]

4. Let $G$ be a group, and construct the direct product $G \times G$. For each $x \in G$ and $(a, b) \in G \times G$, define $x \cdot (a, b)$ by

$$x \cdot (a, b) = a^{-1}xb.$$  

Prove that with this definition, $G \times G$ acts on $G$. [10 points]

5. Let $G$ be a group of order 88. Prove that $G$ is not simple. Be sure to use complete sentences. [10 points]

6. In $S_4$, the subgroup $N = \{(1), (12)(34), (13)(24), (14)(23)\}$ is normal. [15 points]

(a) List the elements of the group $S_4/N$.

(b) Is $S_4/N$ abelian? Justify your answer.
7. Let $G$ be a group, let $N$ and $H$ be subgroups of $G$, and suppose $N \triangleleft G$. Let

$$NH = \{nh : n \in N, h \in H\}.$$ 

It is a fact that $NH$ is a subgroup of $G$; moreover, $N \triangleleft NH$.
You also proved in a homework assignment that $N \cap H \triangleleft H$.
Prove that $H/(H \cap N) \cong NH/N$.

(Suggestion: Define a function $\theta(h) = Nh$ and use the Fundamental Homomorphism Theorem.)

8. Let $G = \mathbb{Z} \times \mathbb{Z}$, and define $\theta : G \rightarrow \text{Sym}(8)$ by

$$\theta((a, b)) = (123)^a(45)^b.$$ 

(a) Find $\ker \theta$.
(b) Find $\theta(G)$ and list all its elements.
(c) Compute $[G : \ker \theta]$. Justify your answer.

9. Use the Sylow-E Theorem to prove Cauchy’s Theorem.

**Cauchy’s Theorem:** Let $G$ be a finite group, and let $p$ be a prime such that $p \mid |G|$. Then $G$ has an element of order $p$.

XC. Let $G$ be a group of order 105 and $H$ a group of order 98. Suppose $\theta : G \rightarrow H$ is a homomorphism, and assume there exists some $a \in G$ such that $\theta(a) \neq e_H$. Find $|\theta(G)|$ and $|\ker \theta|$.