Name: ____________________________________________________________

Practice Exam 2
Modern Algebra
Friday, October 15, 2010

Remember that in general you cannot assume that multiplication of group elements is commutative. You may not use a calculator on this exam. Show all your work.

1. Consider the set \( S = \{10, 14, 2009, 9, 17, 23, -8, 0\} \). Find the equivalence classes of \( S \) under the equivalence relation of congruence modulo 3. [10 points]

2. Let \( a, b, x, y \in \mathbb{Z} \) and suppose \( a \mid x \) and \( b \mid y \). Prove that \( ab \mid xy \). [10 points]

3. Let \( p \) be prime, let \( a \) and \( b \) be integers, and suppose \( a^2 \equiv b^2 \pmod{p} \). Prove that \( a \equiv b \pmod{p} \) or \( a \equiv -b \pmod{p} \).

(HINT: Recall the lemma that if \( p \mid xy \), then \( p \mid x \) or \( p \mid y \).) [10 points]

4. Find the greatest common divisor of 6331 and 741. Be sure to show your work. [10 points]

5. Give an explicit counterexample to disprove the following statement:
   Let \( a, b, c, d \in \mathbb{Z} \). If \((a, b) = 1\) and \((c, d) = 1\), then \((ac, bd) = 1\). [10 points]

6. On the set \( \{x \in \mathbb{R} : x \neq 0\} \), define a relation by saying
   \[ a \sim b \text{ iff } \frac{a}{b} \in \mathbb{Q}, \]
   where as usual \( \mathbb{Q} \) denotes the set of rational numbers. For example, \( 3\pi \sim 7\pi \) since \( \frac{3\pi}{7\pi} = \frac{3}{7} \in \mathbb{Q} \). Prove that \( \sim \) is an equivalence relation. [10 points]

7. Consider the set of positive integers \( \mathbb{N} \). If \( a, b \in \mathbb{N} \), then we can factor them uniquely as
   \[ a = 2^e m \quad \text{ and } \quad b = 2^f n, \]
   where \( e \geq 0, f \geq 0, \) and \( m \) and \( n \) are odd. Define an equivalence relation on \( \mathbb{N} \) by saying \( a \sim b \) if \( e = f \) in that factorization.
   Let \([x]\) denote the equivalence class of \( x \) under \( \sim \), and let \( S \) be the set of all equivalence classes. [10 points]

(a) If we try to define an operation \( \circ \) on \( S \) by saying \([x] \circ [y] = [xy]\), is this “operation” well-defined? If so, prove it; if not, give a specific counterexample with actual numbers.
(b) If we try to define an operation $\oplus$ on $S$ by saying $[x] \oplus [y] = [x + y]$, is this “operation” well-defined? If so, prove it; if not, give a specific counterexample with actual numbers.

8. Let $A$ and $B$ be groups, let $a \in A$, and let $b \in B$. Consider the element $(a, b) \in A \times B$. Prove that $o((a, b)) = o(a) o(b)$. [10 points]

9. In the group $S_5 \times \mathbb{Z}_3$, find the order of the element $((125)(34), [1])$. [10 points]

10. Here is the Cayley table for $D_8$:

<table>
<thead>
<tr>
<th></th>
<th>$R_0$</th>
<th>$R_{90}$</th>
<th>$R_{180}$</th>
<th>$R_{270}$</th>
<th>$H$</th>
<th>$V$</th>
<th>$D_1$</th>
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</tr>
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<tbody>
<tr>
<td>$R_0$</td>
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<td>$D_2$</td>
<td>$D_2$</td>
<td>$H$</td>
<td>$D_1$</td>
<td>$V$</td>
<td>$R_{90}$</td>
<td>$R_{270}$</td>
<td>$R_{180}$</td>
<td>$R_0$</td>
</tr>
</tbody>
</table>

Find the subgroup $\langle R_{270}, H \rangle$ and list all of its elements. Be sure to show your work.

XC. Suppose $A$ and $B$ are subgroups of a group $G$, and suppose their union $A \cup B$ is also a subgroup of $G$. Prove that $A \subseteq B$ or $B \subseteq A$. [10 bonus points]