On a Theory of the Casimir Effect

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Abstract

We develop a mathematically precise framework for the Casimir effect. A major role is played by K. Dietz's idea of identifying the Casimir energy as the regularization-independent Ramanujan sum of an asymptotic series. As an illustration, we treat two cases: parallel plates and the sphere. We finally discuss the open problem of the Casimir force for the cube. We propose an Ansatz for the exterior force and argue why it may provide the exact solution, as well as an explanation of the repulsive sign of the force.

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Significant progress on the Casimir effect from the experimental point of view occurred in recent times [1]. In spite of that, several theoretical problems remain, such as a real explanation of the sign of the force in the case of compact regions. The situation is worse with regard to a mathematically precise framework for the effect, due to the cutoff (or regularization) dependence of the energy, a fact emphasized by C. R. Hagen in [2] and somewhat less emphatically by P. Candelas in [3]. The physical reason why divergences occur is well understood [4] and is that the boundaries are treated by quantizing the radiation field with mode functions [5] which are adapted to the type of (classical) boundary conditions (b.c.), e.g., Dirichlet or Neumann. However, real boundaries consist of electrons and ions and such b.c. are not justified except if the particles act collectively in an essentially classical manner [5], which is a priori not the case [4], and our ignorance in dealing with this fact is signalled by divergences.

Divergences are, of course, well-known in field theory, but they arise here in a different way, as explained above. Mathematical physicists, and several theoretical...
physicists agree that a mathematically precise framework to cope with these divergences would be conceptually useful. Such frameworks exist in field theory (see [6, 7] and references given there). A cutoff-free or “finite” theory of the Casimir effect (in the spirit of [7]) was attempted by G. Scharf and one of us (W.W.) in [8]. It requires, however, the use of periodic b.c., which are unphysical in the case of the electromagnetic field.

In this letter we reconsider the problem, developing some ideas of K. Dietz [9] and in a previous paper by G. Scharf and one of us (W.W.) [8]. Consider an electromagnetic field at $T = 0$ enclosed in cavities of identical shape, but made of different materials, the latter providing natural cutoffs for the high-frequency spectrum of zero point modes. The vacuum energy is thus given by

$$E_{\text{vac}} = \frac{\hbar}{2} \sum_\alpha \omega_\alpha C_\alpha(\Lambda),$$

(1)

with $C_\alpha(\Lambda)$ material dependent cutoff functions depending on a variable $\Lambda$ with dimensions of length, which we normalize by

$$C_\alpha(\Lambda)|_{\Lambda=0} = 1.$$

(2)

Since $E_{\text{vac}}$ has dimension $(\text{length})^{-1}$ in natural units, it may be written as an (asymptotic) series

$$E_{\text{vac}} = a_0 L^3 \Lambda^{-4} + a_1 L^2 \Lambda^{-3} + a_2 L \Lambda^{-2} + a_3 \Lambda^{-1} + a_4 L^{-1} + a_5 L^{-2} \Lambda + \ldots ,$$

(3)

where $L$ is a length characterizing the spatial extension of the cavity. Dietz conjectured [9] that, by a theorem of Ramanujan, the $\Lambda$-independent term $a_4 L^{-1}$ in (3) is independent of the regularization, i.e., of the set $\{C_\alpha(\Lambda)\}$ in (1), provided (2) holds. We shall return to this conjecture.

Consider now as in [8] the prototypical example of massless scalar field $A(x)$ satisfying (for definiteness) Dirichlet boundary conditions (b.c.) $A(x) = 0$ for $\vec{x} \in \partial K$, where $\partial K$ is the boundary of a compact region $K$. Then $A(x)$ may be expanded

$$A(x) = \sum_n \frac{1}{\sqrt{2\omega_n}} [a_n u_n(\vec{x}) e^{-i\omega_n x_0} + a_n^+ u_n(\vec{x}) e^{i\omega_n x_0} ],$$

(4)

where $u_n$ are normalized real eigenfunctions of the Laplacian in $K$, satisfying Dirichlet b.c. (discrete spectrum):

$$-\Delta u_n(\vec{x}) = \omega_n^2 u_n(\vec{x}).$$

(5)

The concrete (Fock) representation is now specified by regarding $a_n^+, a_n$ as emission and absorption operators $[a_n, a_m^+] = \delta_{nm}$, and defining the vacuum by

$$a_n \Omega = 0 \ \forall n .$$

(6)
The Hamiltonian density is [8, 10]

\[ H(x) = \frac{1}{2} \left( \frac{\partial}{\partial x} A(x) \right)^2 - \frac{1}{2} A(x) \frac{\partial^2}{\partial x^2} A(x) : \]

\[ + \frac{1}{i} \lim_{y \to x} \frac{\partial^2}{\partial x_0^2} \left\{ D_0^{(+)}(x - y) - D_K^{(+)}(x, y) \right\}, \]  

where

\[ D_K^{(+)}(x_0 - y_0, \vec{x}, \vec{y}) = i \sum_n \frac{1}{2 \omega_n} u_n(\vec{x}) u_n(\vec{y}) e^{-i \omega_n (x_0 - y_0)} , \]  

and

\[ D_0^{(+)}(x) = \frac{i}{(2\pi)^3} \int \frac{d^3 k}{2 |k|} e^{-i (|k| x_0 - \vec{k} \cdot \vec{x})} . \]  

The two dots in (7) denote normal ordering with respect to the emission and absorption operators \( a_n^+ \) and \( a_n \). We now consider two types of cutoff functions, one of them general, satisfying (2), the other special, of type

\[ C_\alpha(\Lambda) = C(\Lambda \omega_\alpha) , \]  

where

\[ C(0) = 1 , \]  

in correspondence with (2). We shall be interested in a particular case of (10), namely

\[ C(k) = e^{-\Lambda k} \quad k \geq 0 . \]  

In terms of the above special choice (12), one may compute from (7) a regularized vacuum density

\[ H_{\text{vac}}(x, \Lambda) = \frac{1}{2} \frac{\partial}{\partial \Lambda} \left\{ \frac{1}{(2\pi)^3} \int d^3 k \ e^{-i[k_0 \tau - \vec{k} \cdot (\vec{x} - \vec{y})]_{\vec{r}=0}} C(k_0) \right\} \]

\[ - \sum_n [u_n(\vec{x})]^2 C(\omega_n) \right\} . \]  

The Casimir pressure will not depend on the choice (12), see later.

Let, now \( L \) be a linear dimension of the compact region \( K \equiv K_L \) and \( M \) a linear dimension of a region \( K_M \), of which \( K_L \) is a subset. Typically, if \( K_L \) is a cube of side \( L \), \( K_M \) is a cube of side \( M > L \) concentric with \( K_L \), and similarly for a sphere or other manifolds. It is correct to impose the same b.c. (e.g. Dirichlet or Neumann) on \( K_M \) in order to define the outer Casimir problem [11] (see also ref. [12]).
fact, previous work on the sphere using the Sommerfeld radiation condition was not
correct, although the results were right, because it did not lead to real eigenvalues
[13]. Define
\[ E_{\text{vac}}(L, \Lambda, M) = E_{\text{vac}}^{\text{inner}}(L, \Lambda) + E_{\text{vac}}^{\text{outer}}(L, \Lambda, M), \]  
with
\[ E_{\text{vac}}^{\text{inner}}(L, \Lambda) = \int_{K_L} d^3x \, H(\vec{x}, \Lambda); \]
\[ E_{\text{vac}}^{\text{outer}}(L, \Lambda, M) = \int_{K_M \setminus K_L} d^3x \, \tilde{H}(\vec{x}, \Lambda), \]
where, if Dirichlet b.c. are imposed on \( K_L \), \( H \) may be taken to be the density
(13) with the \( \{ u_n \} \) defined by Dirichlet b.c. imposed on \( K_L \), and \( \tilde{H} \) is the density
(13) with the \( \{ u_n \} \) defined by Dirichlet b.c. imposed on \( K_L \) and \( K_M \), by definite-
ness. If (1), (2) is adopted, the second sum in (14) refers, then, to the modes \( \omega_n \)
corresponding to the solution of (5) in \( K_M \setminus K_L \), with the above-mentioned b.c.. Sup-
pose that both \( E_{\text{vac}}^{\text{inner}}(L, \Lambda) \) and \( E_{\text{vac}}^{\text{outer}}(L, \Lambda, M) \) have asymptotic series (3), and let
\( E_{\text{vac}}^{\text{inner}}(L) (\equiv a_4^{\text{inner}} / L) \) and \( E_{\text{vac}}^{\text{outer}}(L, M) \) be the corresponding \( \Lambda \)-independent terms. Then the Casimir pressure \( p_C(L) \) (a measurable quantity) is defined by the thermo-
dynamic formulae (zero absolute temperature):
\[ p_C(L) = p_C^{\text{inner}}(L) - p_C^{\text{outer}}(L), \]
where the relative minus sign takes into account that \( p_C^{\text{outer}} \) refers to a normal vec-
tor pointing inwards towards \( K_L \), whereas \( p_C^{\text{inner}} \) refers to a normal vector pointing
outwards, and
\[ p_C^{\text{inner}}(L) = -\frac{\partial E_{\text{vac}}^{\text{inner}}(L)}{\partial V_{\text{inner}}(L) }; \]
\[ p_C^{\text{outer}}(L) = -\lim_{M \to \infty} \frac{\partial E_{\text{vac}}^{\text{outer}}(L, M)}{\partial V_{\text{outer}}(L, M)}. \]
It is important here that \( M \) is fixed, only \( L \) varies. The limit (19) is expected to exist
on the basis of general results on the thermodynamic limit [14]: this was verified
explicitly for the cube.

Our first result in [10] is that the \( \Lambda \)-independent term in (3) coincides with
the cutoff-independent part of the Ramanujan sum of a divergent series of positive
terms, such as (1), with \( C_\alpha(\Lambda) \equiv 1 \) (see [15], p. 318 ff.). Although our complete
mathematical proof was done for parallel plates, we have checked that the same
method works for the cube or the sphere, although the ultimate details have not yet
been written down.

**Theorem** Let the special cutoff function of type (10) satisfy, besides (11a), the
conditions: \( C \) is infinitely differentiable and its derivatives satisfy
\[ \frac{d^k C(x)}{dx^k} \to 0 \quad \text{as} \quad x \to \infty. \]  
(11b)
and
\[ \int_{\infty}^{\infty} \frac{d^k C(x)}{d x^k} dx < \infty. \quad (11c) \]

Then, for Dirichlet (or Neumann) b.c. the \( \Lambda \)-independent term in (3) is the cutoff-independent part of the Ramanujan sum of the divergent series (1) with \( C_{\alpha}(\Lambda) \equiv 1 \), and is therefore RI, i.e., independent of \( C \).

Thus, except essentially for fixing the value of \( C \) at a point (11a), all cutoff functions yield the same result, i.e., we have RI. In particular, although we used the special cutoff (12) in order to obtain concrete results, our proof shows that these same results are independent of the choice (12). Notice that we do not need to assume that \( \frac{d^k C}{d x^k}(0) = 0 \ \forall \ k \geq 1 \) ([16], p.138): this is not only of academic interest, because this assumption excludes, for instance, the very useful choice (12). See also [17] for a nice approach to the subject. Finally, RI justifies the definition of the \( \Lambda \)-independent term in (3) as the Casimir energy physically: it reflects the field theoretic structure of the ground state which is independent of the cavity materials. It is also expected to be the only term in (3) which contributes to the pressure: this was shown in [9] for parallel plates.

We illustrate the present framework with parallel plates, the sphere and the interior of the cube. For parallel plates at distance \( d \), the energy per unit area \( \lim_{L \to \infty} (E_{\text{vac}}/L^2) = \mathcal{E}(d, \Lambda) \) becomes
\[ \mathcal{E}(d, \Lambda) = -\frac{1}{4\pi} \Lambda^{-3} - \frac{1}{2} \frac{\pi^2}{720d^3} + O(\Lambda), \] (20)
leading to the well-known result for the pressure. Notice, however, that even for the parallel plates there remains an inescapable divergence (as \( \Lambda \to 0 \)) in the first term in (20), which is absent only for periodic b.c. [10], explaining the “finite theory” of [8]. For the sphere, using definition (1) (with a general cutoff not of the type (10)) we have obtained the asymptotic series (3):
\[ E_{\text{asym}} = -\frac{a^2}{8\Lambda^3} - \frac{5}{1024\Lambda} + \frac{0.002819}{a} + \cdots , \] (21)
which yields a repulsive force, as expected. It should be remarked that Bender and Milton [18] already supplied (even with greater precision) the numerical figure in the third term of (21). See also [19] for confirmations of this result by different regularization procedures and [20, 21] for a thorough analysis of the Casimir problem for the sphere. In particular, RI for the ball has been shown in [21].

Consider now a cube \( K_L \) of side \( L \) with Dirichlet b.c.. An analytic approach using (12) and the Poisson sum formula yields (3) for the inner Casimir problem ([10], see also [22]) and thus
\[ p_C^{\text{inner}}(L) = \frac{a_4}{3L^4}, \] (22)
with
\[ a_4 = -\frac{1}{32\pi^2} \sum_{\vec{m} \in \mathbb{Z}^3 \setminus \{0\}} |\vec{m}|^{-4} + \frac{3}{64\pi} \sum_{\vec{m} \in \mathbb{Z}^2 \setminus \{0\}} |\vec{m}|^{-3} - \frac{3}{32\pi} \sum_{m \in \mathbb{Z} \setminus \{0\}} m^{-2} \]
\[ = -0.0157322 \ldots , \] (23)
which yields an attractive force (in ref. [23] the authors obtain the above result by numerical methods). The result for the sphere leads us, however, to expect a repulsive force for the cube also, which, by (22), must be entirely due to the outer problem, for which we introduce the following Ansatz: we split the region \( K_M \setminus K_L \) into 26 subregions bounded by the planes containing the faces of the cube. This introduces additional stresses which will be commented upon later, but yields a soluble problem. The 26 subregions composing \( K_M \setminus K_L \) are of three topologically distinct kinds (with both cubes centered in the origin): 1) a rectangular box with two sides \( L \) and one \( M - L \), with multiplicity 6; 2) a rectangular box with two sides \( M - L \) and one \( L \) (multiplicity 12): the contribution of the edges; 3) a cube of sides \( M - L \) (with multiplicity 8): the contribution of the corners. In the thermodynamic limit the contribution of regions of type 1) and 3) to the pressure is zero. Of central importance is the edges contribution:
\[
E_{\text{edges}}(L, M) = -\frac{3L(M - L)^2}{32\pi^2} \sum_{\vec{m} \in \mathbb{Z}^3 \setminus \{0\}} \frac{1}{m_1^2L^2 + m_2^2(L-M)^2 + m_3^2(L-M)^2} \]
\[ + \frac{3L(M - L)}{16\pi} \sum_{\vec{m} \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{m_1^2L^2 + m_2^2(L-M)^2} \left( \frac{1}{4} \right) \]
\[ + \frac{3}{8\pi(M - L)} \sum_{\vec{m} \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{m_1^2 + m_2^2} - \frac{\pi}{8} \left( \frac{1}{L} + \frac{4}{M - L} \right) . \] (24)

Then, using (19) and noticing that in the thermodynamic limit \( dV_3 \to 0 \) \((dV_3/V_3 \propto 1/M)\) so that \( dV_{\text{outer}} \to 6dV_1 + 12dV_2 = 3M(M - 2L)dL \), we obtain
\[
p_{C \text{outer}}(L) = -\frac{3}{16\pi^2} \zeta(4)L^{-4} = -\frac{\pi^2}{480} L^{-4} , \] (25)

which yields in (17) a repulsive contribution, which is seen to dominate, yielding finally a repulsive force
\[
p_C(L) = (-0.005244 + \frac{\pi^2}{480})L^{-4} = 0.015317 L^{-4} . \] (26)

We (strongly) believe that our Ansatz is the exact solution for the cube for the following reason. We have introduced additional b.c. on planes which are extensions
of the cube’s faces to the region $K_M \setminus K_L$. The additional stresses introduced in the outer region by the extra boundaries are, however, parallel to the faces, and thus the Casimir pressure should be insensitive to their inclusion. We have elaborated on this point, and shown that the Casimir pressure does not change upon introduction of extra stresses on planes orthogonal to a system of parallel plates, both in the outer and inner regions. If our conjecture is right, the calculation leading to (24) also provides an explanation for the sign of the force: it is due to a competition between the inner an the outer pressures, in which the latter is positive and larger than the former in absolute value, because the thermodynamic limit selects a set of modes different from the inner ones, with a large positive contribution from the edges. The edges reflect the passage from the infinitely extended parallel plates to a compact region, i.e., by folding. If this folding were smooth, i.e., for any smooth approximation to the cube, it would be accompanied by nonzero curvature. At the other extreme, uniform nonzero curvature, we have the sphere. Here curvature effects appear less directly, reflecting themselves in the appearance of the Neumann functions in the external problem.

Finally, we come back to the basic problem mentioned at the beginning: we have introduced a mathematically precise framework for the Casimir pressure, by associating it to the Ramanujan sum of the (divergent) series (3).

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References


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   Cambridge, 1982).


