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Educational Systems (ES) in Mathematics and other related topics☆

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ABSTRACT

This work is specifically concerned with various Educational Systems that are used for the improvement of Education in Mathematics. It is based on the speaker's as well as of others' viewpoint which has arisen from discussions about this issue at various levels and on different occasions.

Αγαπητοί Συνάδελφοι και Φίλοι
Honorable guests and participants
Ladies and Gentlemen

Though the official language of the Conference is English, I would like to address my thanks to the Organizers in Greek.

Ευχαριστώ τον Πρόεδρο και τα μέλη της Οργανωτικής Επιτροπής του Συνεδρίου για την πρόσκληση όπως λάβω μέρος στο συνέδριο και να απευθύνω την εναρκτήρια αυτή ομιλία.

Μεταξύ των οργανωτών και των ακροατών διακρίνω παλαιούς μου φοιτητές οι οποίοι τώρα κατέχουν καθηγητικές βαθμίδες καθώς και διάφορα άλλα αξιώματα. Η διαπίστωση αυτή μου προξενεί ένα ιδιαίτερα ευχάριστο συναίσθημα που μοιάζει με «δροσερή αύρα», ένα συναίσθημα, που πολλοί από εσάς, είστε πολύ νέοι για να το νοιώσετε. Εύχομαι σε όλους συνεχή ανοδική επιστημονική πορεία.

Warm congratulations are due to the organizers especially for their idea to include among the aims of the conference to bring together and bequeath scientific activities, directions and pursuits of Greek scientists, in Greece and abroad, on subjects that pertain to the conference.

And now let us come to the subject of my presentation, that is: "Educational Systems (ES) in Mathematics and other related topics", a subject that is supposed to be of general mathematical interest to educators, to students and to anyone else involved in education.

Though many of the viewpoints to be expressed here are known, sporadically published, and shared by many educators, it is useful to repeat them whenever important gatherings of mathematicians, like the present one, occur.

First a short history of the subject.

Most of the ES have treated mathematics as skill in numerical manipulations and have used the terms "quantitative reasoning" to describe the application of mathematics to other areas of study.

This was a serious misconception and it has severely hampered the ability of our students to comprehend important development in scientific and philosophical thought, because the fact, that mathematics is one component of any plan for liberal education was completely ignored in their considerations.

The immediate cause, in this country and elsewhere, of this fact, was the widespread difficulty students apparently experience with this subject.

The reaction of the ES has been an attempt to force students to learn mathematics anyway. The debate was dominated by two conflicting lines of reasoning:

First line: There must be some minimum standards which can be applied to all students.

☆ Invited paper.

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Second line: The dose of mathematics should be limited to what is absolutely necessary. By this logic mathematics courses need be required only as prerequisites for other courses.

In the course of this argument mathematics has been relegated to a position of secondary importance. This implicitly means that mathematics is an esoteric subject, it has some practical applications, but it possesses only tenuous connection to other areas of scholarship.

Any consideration of the intellectual and aesthetic appeal possessed by mathematics has been dismissed as irrelevant for all.

In many curricula, mathematics has been dissociated from other subjects. Physics is nearly always taught as if students know little of mathematics. Similar things can be said about Biology. I heard statisticians say that their subject is something quite apart from mathematics. Some computer-scientists have gone so far to suggest that, if you understand the mathematics of their machines it is not necessary to understand the mathematics of natural phenomena.

The victim here was the student. He or she thinks that what determine whether a course is difficult or easy are the complexity of the problem assignments and the thickness of the textbook. Through course offerings he or she is exposed only to vaguest notions of what we mean by rigor.

A careful consideration of the nature of mathematics, as an intellectual activity, reveals the incongruity of this situation. Not only have our students failed to appreciate the beauty of mathematics, they have little grasp of the profound insights about the natural world which mathematics has made possible. They failed to realize that mathematics is an important facet of the most distinctive capacity of human species.

For the reasons stated above let us take a closer look at the nature and some other aspects of mathematics.

Roughly speaking we can say that mathematics is the study of number and space. To do this study, mathematicians had to invent a language (vocabulary and alphabet). The objects defined are abstracts and are observed only by our imagination.

Because it is abstract, mathematics is comprehensible, and differs from other languages because of its detachment from the complication of what we experience by direct observation.

The inability of existing language to deal with this subject matter was the reason why the language of mathematics was invented. So, Newton invented calculus to express his ideas. Trying to understand Newton without calculus is like trying to understand ΣΟΦΟΚΛΗΣ (Sophocles) and ΑΝΤΙΓΟΝΗ (Antigone) without words.

Euclid's geometry should be appreciated for the beauty which can result from this process. Einstein used to say: "If Euclid failed to kindle your youthful enthusiasm, then you were not born to be a scientific thinker".

Now in experimental science the role of the abstract language of mathematics is that the scientist used "models" (some people call them metaphors) which represent insights into the working of nature. The use of a model is as common place in science as it is in poetry. But the mathematical model is more powerful, than the one in poetry, to uncover implications of the underline idea. By calling the earth a sphere we make a powerful statement because we know many things we can say about spheres.

Another important role that mathematics plays in natural sciences is that it provides to natural sciences a security which cannot be obtained in any other way.

Understanding derives from critical thought not just hard work. Paul Dirac used to say: "I understand what an equation means if I have a way of figuring out the characteristics of its solution without actually solving it".

This is the meaning of the words "physical insight". It is an intuition based on critical analysis of basic questions and the role of the teacher is to "build" insight and not technical dexterity.

So far we have a "vague" picture which represents the role of mathematics in the development of natural sciences.

For the sake of the younger people in this audience let us try to make this picture more clear by examining very briefly some specific areas of mathematics in which application have played an important part. Here are a few examples.

Example 1. Geometry. It began as an empirical science by the Egyptians. The Greeks (culminating in Euclid) made geometry into a postulation system that was a model of mathematical thinking for the next 2000 years.

The change of geometry from an empirical science into a deductive one has changed the course of mathematical history.

During the 18th and 19th centuries attempts to prove Euclid's "parallel axiom" failed. Instead, it was realized that the hyperbolic geometry of Bolyai and Lobachevsky did not lead to a contradiction, that the hyperbolic and the elliptic geometry of Riemann were as logically consistent as Euclidean geometry. They were abstract mathematical systems, but none with a claim as a description of the physical world where we live.

The flow was reversed in the first part of the 20th century, precisely at the time mathematicians were completing the abstraction of geometry. Einstein, looking for a basis for his general theory of relativity, found it in the geometry of Riemann.

The idea that the physical space is finite but unbounded is a cliché of modern day physics. Einstein took the revolutionary step of identifying the physical space with a curved non-Euclidean space. By doing this the debt of the natural sciences (namely the relativity) owed to pure geometry has been repaid. For, many of the ideas of the subject of differential geometry received their initial stimulus from concepts arising from the theory of general relativity. Among these were: manifolds, tangent spaces and some topics from complex geometry.

Example 2. Analysis (that is calculus and the whole group of ideas that followed it) is one of the cornerstones of modern mathematics. It began in the 16th and 17th centuries by trying to develop a theory to account for the observations and measurements, of various phenomena, made by Galileo, Kepler and others.

Newton's **Principia** (1687) stated in mathematical form some physical principles. His law of universal gravitational attraction, together with the laws of motion, became the foundation of mechanics.

In developing these laws, Newton invented his "calculus of fluxions" as a language (tool). We know that the concepts of **Principia** were accepted in the western scientific world and still are used in the study of the behavior of physical systems on earth.

But we also know that for more than 150 years after Newton, there was only a semiphysical formulation of the calculus (inexact).

It was not until Karl Weierstrass (1815–1897) that Analysis became abstract and unempirical.

There is a question: how much of this was because Weierstrass had been a lawyer before he became a mathematician?

Example 3. The area of mathematics known as Harmonic Analysis. In the 19th century when Joseph Fourier (1768–1830) was studying the phenomenon of heat conduction, noticed that trigonometric functions were periodic with different periods and amplitudes, so that linear combinations of them could represent periodic phenomena. To make the long story short it turned out that what he needed was a representation of a function by a Fourier Series (FS). During the 19th century many mathematicians were interested in the study of convergence of FS, and of which functions could be represented by them, and so they were led in surprising directions. For example, Georg Cantor working on FS he found he had to deal with infinite sets of numbers at which an FS converges. From this he was led to study infinitive sets in general and his work on cardinal and ordinal numbers.

Henri Lebesgue noticed that the Riemann's integral was not good enough to compute Fourier coefficients and so he invented the "Lebesgue Integral".

I could give other examples such as Differential Equations (ordinary and partial) – Probability Theory – Group Theory and others, but time is getting short.

We see that there is always a continuous and fruitful interplay between science and mathematics. The science passes through three periods:

(a) The descriptive ones: Call on mathematics and have nothing to give.
(b) The experimental sciences: Use plenty of mathematics and begin to return the investment.
(c) Theoretical sciences: There is a free interplay of ideas with mathematics.

A mathematician's motives are internal: *Intellectual curiosity – sense of form and pattern – taste.*

The above remarks leave the layman perplexed. If professional mathematicians occupy themselves with such things and get paid for doing so what have we gained? These questions give us the opportunity to correct some misapprehensions, and present an invisible part of our culture.

We must reply: so far we have gained nothing. For example the proof of the Fermat Last Theorem (FLT) has no consequences; perhaps even for number theory itself. **But does one ask such kind of question in the face of a masterwork of art, or an impressive achievement in sport?**

Mathematics is, like the arts, a part of our cultural tradition, and has always, in ancient and modern times, obtained its justification from this fact.

On the other hand we should not forget the innumerable applications of mathematics to the physical world.

Now let us see what is happening these days (see Ref. [1], p. 763). We can distinguish two "revolutions" going on in the mathematical, and in general, in science education.

One revolution is how teachers teach and students learn, and the other revolution concerns technology.

As it is the case with all revolutions they carry with them a certain sense of unease.

In the first revolution the debate is among those who advocate a problem-solving approach to mathematics education and in the second revolution are those who believe that "content" (that is theory) should prevail above all else. I think that the ideal lies in between.

Students should learn how to solve problems, but they must also respect that there is a definite body of facts and principles (that is a theory) that guide our thinking at the pursuit of truth.

We should not forget that, what drives most scientists to their work is not the desire to merely ask questions, but to find answers about some part of the world that fascinates them and captures their interests. It is the theory that gives meaning to their research.

The second revolution concerns the set of powerful computers used in many education settings.

Clearly the computers they do not necessarily change the way we learn but they certainly change the speed of how we learn.

Here is a nice example of this phenomenon and gives credit to Numerical Analysis, the subject of today's meeting (see Ref. [2], p. 533).

We know that Schwarz and Christoffel produced (independently) in 1869 the following formula:

$$f(z) = A + B\int_0^z \prod_{k=1}^{n}(1 - \zeta/z_k)^{-\beta_k}\mathrm{d}\zeta. \tag{1}$$

They proved that any conformal map $f(z)$ of the unit disk or of the upper-half plane onto a polygon P with n vertices can be written in the form (1) for some constants A, B, $\langle z_k\rangle$ and $\langle \beta_k\rangle$.

Three computational difficulties arise here for (1).

- Finding the unknown parameters.
- Evaluating the integrals.
- Computing the inverse map.

Analytically (that is without the use of computers) one can do almost nothing to overcome these difficulties. The difficulties must be crossed numerically, using Numerical Analysis and computers. The history here is long. In the past twenty years, because of new algorithms and new computers the S-C conformal mapping of polygons has matured to a technology that can be used **at the touch of a button.**

By contrast with this achievement let me mention a result obtained by Leonhard Euler, whose 300th birthday we are celebrating this year throughout the world.

Euler has proved that the following Fermat's conjecture:

"All natural numbers of the form $2^{2^n}+1$ are prime numbers" is not true by showing that for $n=5$ we have:

$$2^{2^5}+1=2^{32}+1=4,294,967,297=6700\,417\times 641.$$

No need to say that at that time there were no computers around.

When the two revolutions continue to flow, and they do, they force the educators to reconsider most of the up to now existing educational systems. For example they want to know what the effect on students of a computer-based instruction is. They want to know if student-centered learning via the Internet will satisfy their interest that content (that is theory) be mastered.

Some teachers, who use technology, tell us that the introduction of technology into their classrooms has produced sometimes painful transition in their teaching. There are also some other "positive" or "negative" examples of "integrating technology" in teaching.

Much of the debate that has arisen with regard to these two revolutions seems to have the tendency of how to reconcile them. Mathematics instruction seems to face a more complicated situation than do the sciences.

Naturally teaching is an extremely important issue even in the most sophisticated and research intensive departments. It is absolutely necessary to study the teaching-problem in depth, because if it is not solved the debate in question is useless.

I strongly agree that faculty is held to too great a responsibility for what the student learns.

The teacher can present the material but it is up to the student to do the learning.

But the more the responsibility is laid on the student, the more the course has been deemed a success, whatever the method of teaching is.

The university instruction (especially the research) should be based on **discovery learning** guided by mentoring, rather than on the **transmission of knowledge.**

Example (personal experience).

I had to teach the "mean value theorem". So I did, and then I asked the audience if there were questions. There was one: "What was the motivation behind this statement?" My answer was the following: Suppose you drive a car from Patras to Athens and the average speed is 50 k/h. Is it true that at some instant the car was moving with 50 k/h? Of course it is true, says the student, because if the car was moving always with speed greater of less than 50 k/h the average speed would not be 50 k/h. Well, I said, you just stated the mean value theorem in its elementary form.

I recall the saying of a colleague of mine:

The motivation for the approach that the instruction should be based on "discovery-learning" comes from the observation that "the teacher who puts his hand on your shoulder is the one who has had an impact on your life".

Guiding a young researcher in his work (Ph.D. thesis, and other) is also a very delicate kind of teaching. Working hard on problems with no success guaranteed is indeed a big issue. Stubbornness is important but one must know when to give up also. This is true not only for the young researcher but it is true for mature mathematicians.

And now I think it is the right time and place to ask the following standard and typical question asked often in interviews by reporters.

"What is the significance of computers in mathematics? Is it mainly checking experimentally certain conjectures or is it completing proofs by checking an enormous amount of special cases?".

To be honest with you I am not in a position to give a complete answer to this question. I know there are people who think that there are proofs given by computer. I know that computers were used for the four-color problem. Though there are people who say that to check if the four-color solution is correct you have to do it using again computers. Anyway the four-color problem is considered as solved. But we should not blame always the computer. It is the same with the classification of simple group problem.

Coming back to the ES, the challenge for mathematics educators is to find the mid-way between theory and problem solving, to ensure that we reach the desired ES for our children—scientific literacy and finely tuned analytical skills.

The following remarks do not require further explanation.

(a) Everyone can learn the language of mathematics. Mathematics is for all who seek to be truly educated. A mathematical truth is by itself neither simple nor complicated. It just is.

However one must first learn to read, before trying to study the literature of mathematics. This point is systematically violated especially in engineering and science curricula. The students are often confronted by unintelligible mathematics. This situation can be avoided if students were given the chance to study enough mathematics.

(b) Knowing the formal definition of a large number of terms (== quadratic equations – logarithms – Bessel functions, etc.) is no guarantee that one can actually say anything comprehensible. With language, practice is necessary in expressing complete thought. In mathematics this means constructing examples which deal with special objects and events, that is applications.

(c) We train our students to become proficient in answering questions they will never be asked again. The harm we inflict by this sort of skill, since they never really understood the meaning of the answer they gave, they will be unable to answer questions that do come up. The student should not perceive mathematics as a collection of specialized skill, because if we allow this to happen, we risk to produce a legion of mathematical sophomores which possess an extensive but superficial knowledge.

Breaking this cycle is one of the most important pedagogical challenges in mathematics education, whose future is related to the future of mathematics.

(d) Another point that should be understood especially by the teacher is that mathematical exercises should always be chosen in such a way as to provide the student with a keener insight into great concepts, not just practice in manipulations.

Complicated is not the same thing as sophisticated.

The ability to read Greek language does not imply knowledge of the plays of ΣΟΦΟΚΛΗΣ (Sophocles) and ΑΙΣΧΥΛΟΣ (Aeschylus), nor is an understanding of the playwright's ideas derived simply from hearing these words.

Education is not just learning to read and write.

(e) The accomplishments of the past cannot be dismissed. One who is indifferent to the past does not have a future! There is no time to elaborate more on this point. The dismissal of the past is related to the fact that mathematics is categorized only with Science rather than with philosophy and art too. I would argue that mathematics is musical and that it has the characteristics of an art.

But time is pressing.

The arguments and assertions given above imply that our current approaches are ill-conceived and require radical revision. They call for changes in the curriculum. It is envisioned that students will:

1. learn to value mathematics
2. become confident in their own ability
3. become mathematical problem solvers
4. learn to communicate mathematically
5. learn to reason mathematically.

Each of these goals should be elaborated on, the emphasis being on understanding mathematics rather than thoughtlessly grinding out answers.

Clearly this is a very difficult task to accomplish, for there are various concerns leading to resistance to these changes. The reform movements need to address these issues. For instance, one of the concerns is that laudable focus on understanding might lead to some decline in mathematical skills. **Since it is easier to measure and spot deficiencies in skill than understanding,** this problem can easily be overemphasized.

On the other hand this is a serious problem, especially since our future scientists, engineers and mathematicians must obtain both substantial understanding and substantial skills.

Dear friends let me finish with the following concluding words:

Teaching is equally important as research, and furthermore it is great fun.

During my teaching years at the University I put great efforts into teaching and took pride when I was doing a good job. Each of my courses was a ritual, it was a special event for me.

Some mathematicians, who mainly do research, consider teaching as an "unpleasant" duty. I disagree! In years to come computers will be doing more "original" research, which means that the importance of teaching will increase. Moreover, programming a computer is a "teaching" too. In other words teaching is the future.

KEEP TRYING AND BE OPTIMISTIC!

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