17.3 The Divergence Theorem

1. Motivation:

2. Divergence of a vector function:

3. Calculate the divergence of $\mathbf{F} = \langle \boldsymbol{e^x \sin y,\ e^x \cos y,\ z} \rangle$.

4. Divergence Theorem: Let $\boldsymbol{\mathcal{W}}$ be a simple solid region in $\mathbb{R}^{\mathbf{3}}$ and $\boldsymbol{\mathcal{S} = \partial\mathcal{W}}$ be the boundary surface of $\boldsymbol{\mathcal{W}}$ with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous first-order partial derivatives. Then

$$\iint_{\boldsymbol{\mathcal{S}}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\boldsymbol{\mathcal{W}}} \mathrm{div}(\mathbf{F})\, dV.$$

5. Discussion:

6. Let $\boldsymbol{\mathcal{W}}$ be the solid unit ball in $\mathbb{R}^{\mathbf{3}}$. The boundary surface $\boldsymbol{\mathcal{S}} = \boldsymbol{\partial\mathcal{W}}$ is the unit sphere

$$\boldsymbol{x^2 + y^2 + z^2 = 1}.$$

Verify the Divergence Theorem for the vector field $\mathbf{F} = \langle \boldsymbol{3x, 3y, 3z} \rangle$.

7. Let $\boldsymbol{\mathcal{W}} = \boldsymbol{\{(x, y, z) : \ x, y, z \in (-1, 1)\}}$ be the cube centered at the origin with faces parallel to the coordinate planes and side length 2. Verify the Divergence Theorem for the vector field $\mathbf{F} = \langle \boldsymbol{2x^2, \ -y^2, \ z} \rangle$.

8. Use the Divergence Theorem to calculate the flux of the vector field $\mathbf{F} = \langle -x^3,\ -y^3,\ 3z^2 \rangle$ across the boundary $\boldsymbol{\mathcal{S}}$ of

$$\boldsymbol{\mathcal{W}} = \{(x, y, z) :\ x^2 + y^2 < 16,\ 0 < z < 5\}.$$

9. Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = \left\langle \boldsymbol{xy},\ \boldsymbol{y^2 + e^{xz^2}},\ \mathbf{sin}(\boldsymbol{xy}) \right\rangle$$

and $\boldsymbol{\mathcal{S}}$ is the surface of the region $\boldsymbol{\mathcal{W}}$ bounded by the parabolic cylinder

$$\boldsymbol{z = 1 - x^2}$$

and the planes $\boldsymbol{z = 0}$, $\boldsymbol{y = 0}$, $\boldsymbol{y + z = 2}$.