

17.1 Green's Theorem

1. Circulation—a line integral of a vector field around a closed curve. What is the circulation of a gradient vector field?
2. Suppose \mathcal{C} is a simple closed curve that forms the boundary of a region \mathcal{D} in the xy -plane. Let \mathcal{C} be oriented so that traversing \mathcal{C} in the positive direction keeps \mathcal{D} to the left, i.e. counterclockwise orientation. The corresponding line integral of $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ around \mathcal{C} is denoted by $\oint_{\mathcal{C}} P(x, y) \, dx + Q(x, y) \, dy$.
3. Green's Theorem: Let \mathcal{D} be a domain whose boundary $\mathcal{C} = \partial\mathcal{D}$ is a simple closed piecewise smooth curve in the plane, positively oriented (counterclockwise). If $\mathbf{F} = \langle P, Q \rangle$, where $P(x, y)$ and $Q(x, y)$ are continuous and have continuous partial derivatives, then

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial\mathcal{D}} P \, dx + Q \, dy = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

4. Evaluate $\oint_{\mathcal{C}} x^2 y \, dx + x \, dy$ using Green's Theorem, and then check the answer by evaluating the line integral directly, where \mathcal{C} is the triangle connecting the points $(0, 0)$, $(1, 0)$, and $(1, 2)$, oriented counterclockwise.

5. Note: For $\mathbf{F} = \langle P, Q \rangle$, the scalar curl is $\text{curl}_z(\mathbf{F}) = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. Thus Green's Theorem can be written as

$$\oint_{\partial \mathcal{D}} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial \mathcal{D}} P \, dx + Q \, dy = \iint_{\mathcal{D}} \text{curl}_z(\mathbf{F}) \, dA.$$

6. Find the work done by the force field $\mathbf{F} = \langle e^x - y^3, \cos y + x^3 \rangle$ on a particle traveling around the unit circle, counterclockwise.

7. Area:

8. Use a line integral to find the area of the region enclosed by the astroid

$$x = a \cos^3 t, \quad y = a \sin^3 t, \quad 0 \leq t \leq 2\pi.$$

9. Use Green's Theorem to evaluate $\oint_{\mathcal{C}} \ln(1 + y) \, dx - \frac{xy}{1 + y} \, dy$, where \mathcal{C} is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$, positively oriented.

10. Find the line integral of $\mathbf{F} = \langle x^2 + y, 4x - \cos y \rangle$ around the boundary of the region \mathcal{D} that is inside the square with vertices $(0, 0)$, $(5, 0)$, $(5, 5)$, and $(0, 5)$, but is outside the rectangle with vertices $(1, 1)$, $(3, 1)$, $(3, 2)$, and $(1, 2)$. Assume the boundary of \mathcal{D} is oriented such that \mathcal{D} is on the left when the boundary is traversed.