16.5 Surface Integrals of Vector Fields

1. Motivation/Interpretation:

2. Oriented Surfaces:

3. Derivation:

4. Definition: Given a vector field $\mathbf{F}$ with unit normal vector $\mathbf{e_n}$, the surface integral of $\mathbf{F}$ over the surface $\boldsymbol{\mathcal{S}}$ is

$$\iint_{\boldsymbol{\mathcal{S}}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\boldsymbol{\mathcal{S}}} (\mathbf{F} \cdot \mathbf{e_n})\, dS.$$

   This integral is also called the flux of $\mathbf{F}$ across the surface $\boldsymbol{\mathcal{S}}$.

5. Note: If the surface $\boldsymbol{\mathcal{S}}$ is given as the parametrized surface

$$\boldsymbol{\Phi(u,v) = \Big(x(u,v),\ y(u,v),\ z(u,v)\Big)},$$

   then

$$\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v, \quad \text{so} \quad \mathbf{e_n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{\|\mathbf{T}_u \times \mathbf{T}_v\|}.$$

   Once we find $\mathbf{e_n}$, look at it to determine if it is the appropriate orientation; if not, use $-\mathbf{e_n}$.

6. Recall from Section 16.4 that $dS = \|\mathbf{n}\|\, dA$, and thus

$$\begin{aligned}
\iint_{\boldsymbol{\mathcal{S}}} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\boldsymbol{\mathcal{S}}} (\mathbf{F} \cdot \mathbf{e_n})\, dS \\
&= \iint_{\boldsymbol{\mathcal{D}}} \mathbf{F}(\boldsymbol{\Phi}(\boldsymbol{u}, \boldsymbol{v})) \cdot \frac{\mathbf{n}}{\|\mathbf{n}\|} \|\mathbf{n}\|\, dA \\
&= \iint_{\boldsymbol{\mathcal{D}}} \mathbf{F}(\boldsymbol{\Phi}(\boldsymbol{u}, \boldsymbol{v})) \cdot \mathbf{n}(\boldsymbol{u}, \boldsymbol{v})\, du dv.
\end{aligned}$$

7. Let $\mathbf{F} = \langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle$ over the portion $\boldsymbol{\mathcal{S}}$ of the surface $\boldsymbol{z = 1 - x^2 - y^2}$ that lies above the $\boldsymbol{xy}$-plane, with upward-pointing normal. Determine the value of the surface integral of $\mathbf{F}$ over $\boldsymbol{\mathcal{S}}$.

8. Find the flux of $\mathbf{F} = \langle \boldsymbol{z}, \boldsymbol{y}, \boldsymbol{x} \rangle$ across the unit sphere $\boldsymbol{x^2 + y^2 + z^2 = 1}$, oriented outward.

9. Let $\mathbf{v} = \langle \boldsymbol{2x},\ \boldsymbol{-3y},\ \boldsymbol{z} \rangle$ be the velocity field, in ft/s, of a fluid in $\mathbb{R}^{\mathbf{3}}$. Calculate the flux, in ft^3/s, through the portion of the plane $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} = \mathbf{1}$ in the first octant.