16.3 Conservative Vector Fields

1. Let $\boldsymbol{\varphi(x, y) = xy^3 + x^2}$. Compute the work done along the curve $\boldsymbol{\mathcal{C}}$ by the gradient vector field of $\boldsymbol{\varphi}$ (see picture).

2. **Fundamental Theorem for Line Integrals**

3. Note: For a closed curve $\boldsymbol{\mathcal{C}}$, $\mathbf{F} = \boldsymbol{\nabla\varphi}$ implies that $\displaystyle\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \mathbf{0}$.

4. Evaluate $\displaystyle\int_{\mathcal{C}} \boldsymbol{\nabla\varphi} \cdot d\mathbf{s}$, where $\boldsymbol{\varphi(x, y, z) = \cos(\pi x) + \sin(\pi y) - xyz}$ for any curve $\boldsymbol{\mathcal{C}}$ from $\boldsymbol{(1, \frac{1}{2}, 2)}$ to $\boldsymbol{(2, 1, -1)}$.

5. **Equivalent Conditions for Path Independence**

Assume $\mathbf{F}$ is a continuous vector field in some domain $\mathcal{D}$.

- $\mathbf{F}$ is a conservative vector field if there is a function φ such that $\mathbf{F} = \nabla\varphi$. The function φ is called a potential function for the vector field.
- $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ is independent of path if $\displaystyle\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{s}$ for any two paths $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{D}$ with the same initial and final points.
- A path $\mathcal{C}$ is closed if its initial and final points coincide. For example, any circle is a closed path.
- A path $\mathcal{C}$ is simple if it does not cross itself. For example, a circle is a simple curve, but a figure 8 is not simple.
- A region $\mathcal{D}$ is open if it does not contain any of its boundary points.
- A region $\mathcal{D}$ is connected if we can connect any two points in the region with a path that lies completely in $\mathcal{D}$.
- A region $\mathcal{D}$ is simply connected if it is connected and it contains no holes.

6. Conservative Vector Fields: Given a vector field, when is it conservative? If it is conservative, what is the potential?

7. Existence of a Potential Function: Let $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ be a vector field on a simply-connected domain $\mathcal{D}$. If the cross partials of $\mathbf{F}$ are equal, that is

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y},$$

then $\mathbf{F} = \nabla\varphi$ for some potential φ.

8. Determine if the following vector fields are conservative and find a potential function for the vector field if it is conservative.

   (a) $\mathbf{F} = \langle y + x,\ y - x \rangle$

   (b) $\mathbf{F} = \langle \sin y + y \sin x,\ 1 + x \cos y - \cos x \rangle$

9. Show that the vector field $\mathbf{F} = \langle \boldsymbol{yz^2},\ \boldsymbol{xz^2 - z},\ \boldsymbol{2xyz - y} \rangle$ is conservative and find a potential function for it.

10. Show that the vector field $\mathbf{F} = \langle \boldsymbol{z^3},\ \boldsymbol{-2yz},\ \boldsymbol{3xz^2 - y^2} \rangle$ is conservative and find a potential function for it.

11. Vortex Vector Field: Consider $\mathbf{F} = \left\langle \dfrac{-\boldsymbol{y}}{\boldsymbol{x}^2 + \boldsymbol{y}^2}, \dfrac{\boldsymbol{x}}{\boldsymbol{x}^2 + \boldsymbol{y}^2} \right\rangle$.