16.2 Vector Line Integrals

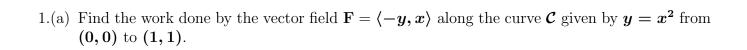
- The (scalar) line integral of a function f(x, y, z) along a curve C is denoted by $\int_{C} f(x, y, z) ds$.
- Computing a Scalar Line Integral: Let c(t) = (x(t), y(t), z(t)) be a path parametrization of a curve \mathcal{C} for $a \leq t \leq b$. Assume that f(x, y, z) and c'(t) are continuous. Then

$$\int_{\mathcal{C}} f(x, y, z) ds = \int_{a}^{b} f(c(t)) ||c'(t)|| dt$$

$$= \int_{a}^{b} f(x(t), y(t), z(t)) \sqrt{(x'(t))^{2} + (y'(t))^{2} + (z'(t))^{2}} dt.$$

The value of the integral on the right is independent of the parametrization.

• Work done by a vector field



1.(b) Another way (notation change)

- 2. (From MIT OpenCourseWare) Find the work done by the electrostatic field $\mathbf{F}=\langle y,z,x\rangle$ in carrying a positive unit point charge from (1,1,1) to (2,4,8) along
 - (a) a straight line segment; (b) the curve described by $c(t)=(t,t^2,t^3)$.

- The work W done by the vector field F along the curve C is given by $W = \int_{C} F \cdot ds$.
- Let \mathcal{C} be an oriented curve, and let \mathbf{T} be the unit tangent vector pointing in the forward direction along \mathcal{C} . The (vector) line integral of a vector field \mathbf{F} along \mathcal{C} is the integral of the tangential component of \mathbf{F} :

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{T}) \ ds.$$

• Computing a Vector Line Integral: Let c(t) = (x(t), y(t), z(t)) be a regular path parametrization of an oriented curve \mathcal{C} for $a \leq t \leq b$; here regular means $c'(t) \neq 0$. Then the line integral of a vector field $F = \langle F_1, F_2, F_3 \rangle$ over the curve \mathcal{C} is

$$\int_{c} \mathbf{F} \cdot d\mathbf{s} = \int_{a}^{b} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

$$= \int_{a}^{b} \mathbf{F}(x(t), y(t), z(t)) \cdot \langle x'(t), y'(t), z'(t) \rangle dt.$$

This can also be written as

$$\int_{\mathcal{C}} F_1 dx + F_2 dy + F_3 dz = \int_a^b \left(F_1(\mathbf{c}(t)) \frac{dx}{dt} + F_2(\mathbf{c}(t)) \frac{dy}{dt} + F_3(\mathbf{c}(t)) \frac{dz}{dt} \right) dt.$$

- Properties of Line Integrals: Let \mathcal{C} be a smooth oriented curve, and let \mathbf{F} and \mathbf{G} be vector fields.
 - (i) Linearity: $\int_{\mathcal{C}} (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{s} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{s}$ and $\int_{\mathcal{C}} k\mathbf{F} \cdot d\mathbf{s} = k \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$
 - (ii) Reverse Orientation: $\int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$
 - (iii) Additivity: If \mathcal{C} is a union of n smooth curves $\mathcal{C}_1 + \cdots + \mathcal{C}_n$, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \dots + \int_{\mathcal{C}_n} \mathbf{F} \cdot d\mathbf{s}.$$

• The value of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ depends on the curve (trajectory) \mathcal{C} , but not on the parametrization of \mathcal{C} .