16.2 Vector Line Integrals

- The (scalar) line integral of a function $f(x, y, z)$ along a curve $\mathcal{C}$ is denoted by $\int_{\mathcal{C}} f(x, y, z)\, ds$.

- **Computing a Scalar Line Integral**: Let $\mathbf{c}(t) = \big(x(t), y(t), z(t)\big)$ be a path parametrization of a curve $\mathcal{C}$ for $a \le t \le b$. Assume that $f(x, y, z)$ and $\mathbf{c}'(t)$ are continuous. Then

$$\begin{aligned}\int_{\mathcal{C}} f(x, y, z)\, ds &= \int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\|\, dt \\ &= \int_a^b f\big(x(t), y(t), z(t)\big) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.\end{aligned}$$

  The value of the integral on the right is independent of the parametrization.

- Work done by a vector field

1.(a) Find the work done by the vector field $\mathbf{F} = \langle -\boldsymbol{y}, \boldsymbol{x} \rangle$ along the curve $\boldsymbol{\mathcal{C}}$ given by $\boldsymbol{y} = \boldsymbol{x}^{\mathbf{2}}$ from $\mathbf{(0, 0)}$ to $\mathbf{(1, 1)}$.

1.(b) Another way (notation change)

2. (From MIT OpenCourseWare) Find the work done by the electrostatic field $\mathbf{F} = \langle \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{x} \rangle$ in carrying a positive unit point charge from $\mathbf{(1, 1, 1)}$ to $\mathbf{(2, 4, 8)}$ along

(a) a straight line segment; (b) the curve described by $\boldsymbol{c}(\boldsymbol{t}) = (\boldsymbol{t}, \boldsymbol{t}^{\mathbf{2}}, \boldsymbol{t}^{\mathbf{3}})$.

- The work W done by the vector field $\mathbf{F}$ along the curve $\mathcal{C}$ is given by $W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$.

- Let $\mathcal{C}$ be an oriented curve, and let $\mathbf{T}$ be the unit tangent vector pointing in the forward direction along $\mathcal{C}$. The (vector) line integral of a vector field $\mathbf{F}$ along $\mathcal{C}$ is the integral of the tangential component of $\mathbf{F}$:
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{T})\, ds.$$

- **Computing a Vector Line Integral**: Let $\mathbf{c}(t) = \big(x(t), y(t), z(t)\big)$ be a regular path parametrization of an oriented curve $\mathcal{C}$ for $a \le t \le b$; here regular means $\mathbf{c}'(t) \neq \mathbf{0}$. Then the line integral of a vector field $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ over the curve $\mathcal{C}$ is
$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \\ &= \int_a^b \mathbf{F}\big(x(t), y(t), z(t)\big) \cdot \langle x'(t), y'(t), z'(t) \rangle\, dt. \end{aligned}$$
This can also be written as
$$\int_{\mathcal{C}} F_1\, dx + F_2\, dy + F_3\, dz = \int_a^b \left(F_1(\mathbf{c}(t))\frac{dx}{dt} + F_2(\mathbf{c}(t))\frac{dy}{dt} + F_3(\mathbf{c}(t))\frac{dz}{dt} \right) dt.$$

- **Properties of Line Integrals**: Let $\mathcal{C}$ be a smooth oriented curve, and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields.
  - (i) Linearity: $\int_{\mathcal{C}} (\mathbf{F} + \mathbf{G}) \cdot d\mathbf{s} = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{s}$ and $\int_{\mathcal{C}} k\mathbf{F} \cdot d\mathbf{s} = k \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$
  - (ii) Reverse Orientation: $\int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$
  - (iii) Additivity: If $\mathcal{C}$ is a union of n smooth curves $\mathcal{C}_1 + \cdots + \mathcal{C}_n$, then
$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{s} + \cdots + \int_{\mathcal{C}_n} \mathbf{F} \cdot d\mathbf{s}.$$

- The value of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}$ depends on the curve (trajectory) $\mathcal{C}$, but not on the parametrization of $\mathcal{C}$.