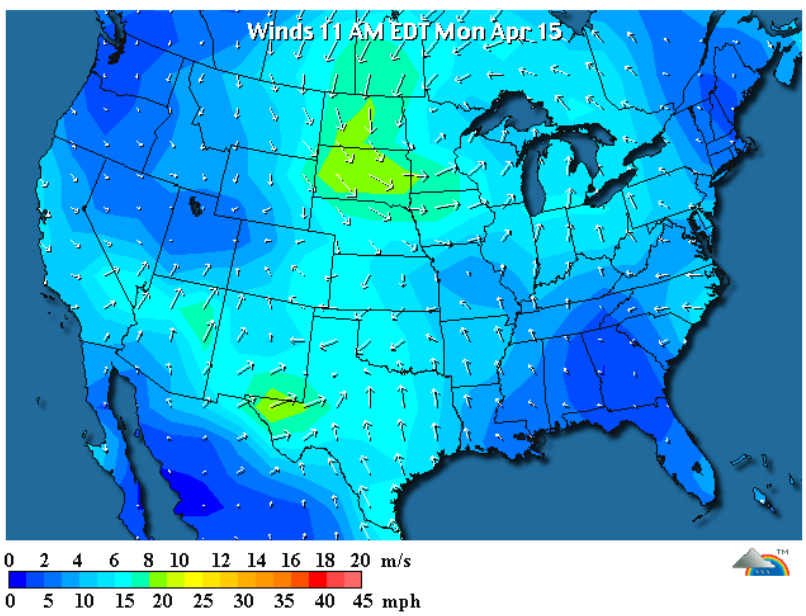
16.1 Vector Fields

- A vector field in $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) a vector $\mathbf{F}(x, y)$, or in $\mathbb{R}^3$ to each point (x, y, z) a vector $\mathbf{F}(x, y, z)$.
- *Mathematica* examples: $\mathbf{F}_1(x, y) = \mathbf{i} - \mathbf{j}$; $\mathbf{F}_2(x, y) = \langle y, -x \rangle$; $\mathbf{F}_3(x, y, z) = \langle x, y, z \rangle$; $\mathbf{F}_4(x, y, z) = \langle x, -y, \cos x \rangle$;
- For a function $\varphi(x, y, z)$ the gradient vector is defined by $\mathbf{F} = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z \rangle$.

  This $\mathbf{F}$ is often called a gradient vector field, and the function φ is the potential function for $\mathbf{F}$.
- Wind velocity vector field



1. Unit Vector Field: $\mathbf{F}$ is a unit vector field iff $\|\mathbf{F}(\boldsymbol{P})\| = \mathbf{1}$ for all points $\boldsymbol{P}$. The unit radial vector fields are

$$\boldsymbol{e_r} = \left\langle \frac{\boldsymbol{x}}{\boldsymbol{r}}, \frac{\boldsymbol{y}}{\boldsymbol{r}} \right\rangle \quad \text{and} \quad \boldsymbol{e_r} = \left\langle \frac{\boldsymbol{x}}{\boldsymbol{r}}, \frac{\boldsymbol{y}}{\boldsymbol{r}}, \frac{\boldsymbol{z}}{\boldsymbol{r}} \right\rangle,$$

where $\boldsymbol{r} = (\boldsymbol{x}^2 + \boldsymbol{y}^2)^{1/2}$ in $\mathbb{R}^2$ and $\boldsymbol{r} = (\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2)^{1/2}$ in $\mathbb{R}^3$, respectively.

2. Find the gradient vector field for $\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^2 \sin(\boldsymbol{\pi y})$

3. Find the gradient vector field for $\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{z e^{-xy}}$

4. Sketch the gradient vector field for $\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^2\boldsymbol{y} - \boldsymbol{y}^3$, together with several contours for this function (*Mathematica*).

5. Cross Partials: The cross partials of a gradient vector field are equal.

16.2 Line Integrals for Scalar Functions

- The line integral of a function $f(x, y, z)$ over a curve $\mathcal{C}$ is called a scalar line integral and is denoted by $\int_{\mathcal{C}} f(x, y, z)\, ds$.
- Theorem: Let $\mathbf{c}(t) = \big(x(t), y(t), z(t)\big)$ be a path parametrization of a curve $\mathcal{C}$ for $a \le t \le b$. Assume that $f(x, y, z)$ and $\mathbf{c}'(t)$ are continuous. Then

$$\begin{aligned}\int_{\mathcal{C}} f(x, y, z)\, ds &= \int_a^b f(\mathbf{c}(t))\|\mathbf{c}'(t)\|\, dt \\ &= \int_a^b f\big(x(t), y(t), z(t)\big)\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt.\end{aligned}$$

  The value of the integral on the right is independent of the parametrization. For $f(x, y, z) = 1$ we obtain the length of $\mathcal{C}$:

$$\text{Length of } \mathcal{C} = \int_{\mathcal{C}} \|\mathbf{c}'(t)\|\ dt.$$

1. Evaluate the line integral $\int_{\mathcal{C}} (xy + z^3)\, ds$ along the helix $\mathcal{C}$ given by $\mathbf{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi$.