15.4 Double Integrals in Polar Coordinates

- Recall that for a continuous function $\boldsymbol{f(x,y)}$ defined on the region

$$\boldsymbol{R = \{(x,y) : a \le x \le b, g_1(x) \le y \le g_2(x)\}}$$

for continuous functions $\boldsymbol{g_1, g_2}$ with $\boldsymbol{g_1(x) \le g_2(x)}$ for all $\boldsymbol{x \in [a,b]}$, the double integral of $\boldsymbol{f}$ over $\boldsymbol{R}$ is

$$\boldsymbol{\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx},$$

interpreted as the volume under the surface $\boldsymbol{z = f(x,y)}$ over the region $\boldsymbol{R}$.

- Using polar coordinates:

$$\boldsymbol{x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2}$$

we wish to derive the double integral formula

$$\boldsymbol{\iint_R f(x,y)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta}$$

over

$$\boldsymbol{R = \{(r,\theta) : \alpha \le \theta \le \beta, g_1(\theta) \le r \le g_2(\theta)\}}$$

for continuous functions $\boldsymbol{g_1, g_2}$ with $\boldsymbol{g_1(\theta) \le g_2(\theta)}$ for all $\boldsymbol{\theta \in [\alpha, \beta]}$.

- Derivation:

1. Evaluate $\displaystyle\iint_{\boldsymbol{R}} (\boldsymbol{3x + 4y^2})\boldsymbol{dA}$, where $\boldsymbol{R}$ is the region in the upper half plane bounded by the circles $\boldsymbol{x^2 + y^2 = 1}$ and $\boldsymbol{x^2 + y^2 = 4}$.

2. Sketch the region of integration and evaluate $\displaystyle\int_{\mathbf{0}}^{\mathbf{1/2}} \int_{\sqrt{\mathbf{3}}\,\boldsymbol{x}}^{\sqrt{\mathbf{1}-\boldsymbol{x}^2}} \boldsymbol{x\ dy\ dx}$.

3. Integrate $\boldsymbol{f(x,y) = (x^2+y^2)^{-2}}$ over $\boldsymbol{\mathcal{D}}$ given by $\boldsymbol{x \geq 1}$, $\boldsymbol{x^2+y^2 \leq 2}$. Sketch $\boldsymbol{\mathcal{D}}$.

4. Sketch $\boldsymbol{\mathcal{D}}$ and integrate $\boldsymbol{f(x,y)}$, where $\boldsymbol{f(x,y) = y(x^2+y^2)^{-1}}$ and $\boldsymbol{\mathcal{D}}$ is $\boldsymbol{y \geq \frac{1}{2}}$, $\boldsymbol{x^2+y^2 \leq 1}$.

5. Find the volume of the solid that lies under the paraboloid $\boldsymbol{z = x^2 + y^2}$, above the $\boldsymbol{xy}$−plane, and inside the cylinder $\boldsymbol{x^2 + y^2 = 2x}$.