

15.3 Triple Integrals

- Fubini's Theorem: If $f(x, y, z)$ is continuous on the box \mathcal{B} defined by

$$\mathcal{B} = \{(x, y, z) : a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\},$$

then the triple integral of f over \mathcal{B} is

$$\iiint_{\mathcal{B}} f(x, y, z) \, dV = \int_r^s \int_c^d \int_a^b f(x, y, z) \, dx \, dy \, dz,$$

where the integrals are evaluated from the inside out.

- Theorem 2: Let \mathcal{D} be a region in the xy -plane. Assume that $\psi(x, y)$ and $\phi(x, y)$ are continuous with $\psi(x, y) \leq \phi(x, y)$ for $(x, y) \in \mathcal{D}$. Then the triple integral of a continuous function $f(x, y, z)$ over the domain

$$\mathcal{W} = \{(x, y, z) : (x, y) \in \mathcal{D}\} \quad \text{and} \quad \psi(x, y) \leq z \leq \phi(x, y)$$

exists and is equal to the iterated integral

$$\iiint_{\mathcal{W}} f(x, y, z) \, dV = \iint_{\mathcal{D}} \left(\int_{z=\psi(x,y)}^{z=\phi(x,y)} f(x, y, z) \, dz \right) \, dA$$

1. Evaluate the triple integral $\iiint_{\mathcal{B}} xyz^2 \, dV$, where \mathcal{B} is the rectangular box

$$\mathcal{B} = \{(x, y, z) : 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}.$$

Integrate with respect to x , then y , then z .

2. Evaluate $\iiint_{\mathcal{W}} z \, dV$, where \mathcal{W} is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

3. Let \mathcal{W} be the region bounded by

$$z = 4 - y^2, \quad y = 2x, \quad z = 0, \quad x \geq 0.$$

Express $\iiint_{\mathcal{W}} xyz \, dV$ as an iterated integral in three orders:

$$dz \, dx \, dy, \quad dx \, dy \, dz, \quad dy \, dx \, dz.$$

4. Use a triple integral to find the volume of the tetrahedron bounded by the planes $x = 2y$, $x = 0$, $x + 2y + z = 2$, and $z = 0$.

5. Find the volume of the solid in \mathbb{R}^3 bounded by $x = y^2$, $y = x^2$, $z = x + y + 5$, and $z = 0$.

6. Evaluate $\iiint_{\mathcal{W}} \sqrt{x^2 + z^2} \, dV$, where \mathcal{W} is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.