

15.2 Double Integrals over General Regions

1. Theorem: For a continuous function $f(x, y)$ defined on the region

$$R = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

for continuous functions g_1, g_2 with $g_1(x) \leq g_2(x)$ for all $x \in [a, b]$, the double integral of f over R is

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

If the region $R = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$ for continuous functions h_1, h_2 with $h_1(y) \leq h_2(y)$ for all $y \in [c, d]$, the double integral of f over R is

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$

2. Evaluate

$$\int_0^4 \int_{\sqrt{y}}^2 (xy + y) dx dy,$$

then switch the order of integration and evaluate again.

3. Evaluate $\int_0^1 \int_y^1 e^{x^2} dx dy$.

4. Evaluate $\iint_R \mathbf{x}\mathbf{y} \, d\mathbf{A}$ over the region \mathbf{R} enclosed between $\mathbf{y} = \frac{1}{2}\mathbf{x}$, $\mathbf{y} = \sqrt{\mathbf{x}}$, $\mathbf{x} = 2$, and $\mathbf{x} = 4$.

5. Evaluate $\iint_R (2\mathbf{x} - \mathbf{y}^2) \, d\mathbf{A}$ over the triangular region \mathbf{R} enclosed by $\mathbf{y} = 1 - \mathbf{x}$, $\mathbf{y} = 1 + \mathbf{x}$, and $\mathbf{y} = 3$.

6. Compute the integral of the function $f(x, y) = x + y$ for $0 \leq x \leq 3$ and $0 \leq y \leq \sqrt{9 - x^2}$.
7. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the xy plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

8. Average Value: The average value of a function $\mathbf{f}(\mathbf{x}, \mathbf{y})$ on a domain \mathcal{D} is defined by

$$\bar{\mathbf{f}} = \frac{1}{\text{Area}(\mathcal{D})} \iint_{\mathcal{D}} \mathbf{f}(\mathbf{x}, \mathbf{y}) \, d\mathbf{A} = \frac{\iint_{\mathcal{D}} \mathbf{f}(\mathbf{x}, \mathbf{y}) \, d\mathbf{A}}{\iint_{\mathcal{D}} 1 \, d\mathbf{A}}.$$

9. Calculate the average value of the \mathbf{x} -coordinate of a point on the semicircle $\mathbf{x}^2 + \mathbf{y}^2 \leq R^2$, $\mathbf{x} \geq \mathbf{0}$. What is the average value of the \mathbf{y} -coordinate?