15.1 Double Integrals

1. Review of Integration/Area: The definite integral of f over the interval $[a, b]$ is

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

provided the limit exists, and is the same for all choices of $c_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$. We interpret this as representing the area under the curve of f from a to b.

2. Volume: Given a continuous function $f(x, y) \geq 0$ on the rectangle $a \leq x \leq b$, $c \leq y \leq d$, we wish to find the volume of the solid under the surface of $z = f(x, y)$ and above the rectangle in the xy plane.

3. Double Integral: For any function $f(x, y)$ defined on the rectangle

$$R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$$

the double integral of f over R is

$$\begin{aligned} \iint_R f(x, y)dA &= \lim_{\max \Delta A_i \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i \\ &= \lim_{M,N \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij})\Delta x_i \Delta y_j \end{aligned}$$

where $P_{ij} = (x_{ij}, y_{ij})$, provided the limit exists, and is the same for all choices of $(x_{ij}, y_{ij}) \in R$ for $i = 1, \ldots, n$. We interpret this as representing the volume under the surface of f for $a \leq x \leq b$ and $c \leq y \leq d$.

4. Example: Go to page 883, Figure 17, and problem #3.

5. Iterated Integrals: From Chapter 6 (page 382), the volume $\boldsymbol{V}$ is

$$\boldsymbol{V} = \int_a^b \boldsymbol{A(x)dx} \quad \text{or} \quad \boldsymbol{V} = \int_c^d \boldsymbol{A(y)dy}.$$

For fixed $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$ or fixed $\boldsymbol{y} \in [\boldsymbol{c}, \boldsymbol{d}]$, we have

$$\boldsymbol{A(x)} = \int_c^d \boldsymbol{f(x,y)dy} \quad \text{or} \quad \boldsymbol{A(y)} = \int_a^b \boldsymbol{f(x,y)dx}.$$

Therefore the volume $\boldsymbol{V}$ is given by

$$\boldsymbol{V} = \int_a^b \left[\int_c^d \boldsymbol{f(x,y)dy}\right] \boldsymbol{dx} \quad \text{or} \quad \boldsymbol{V} = \int_c^d \left[\int_a^b \boldsymbol{f(x,y)dx}\right] \boldsymbol{dy}.$$

6. Fubini's Theorem: If $\boldsymbol{f}$ is integrable over the rectangle $\boldsymbol{R}$ given by

$$\boldsymbol{R} = \{\boldsymbol{(x,y) : a \le x \le b,\ c \le y \le d}\},$$

then the double integral of $\boldsymbol{f}$ over $\boldsymbol{R}$ is

$$\iint_{\boldsymbol{R}} \boldsymbol{f(x,y)dA} = \int_a^b \left[\int_c^d \boldsymbol{f(x,y)dy}\right] \boldsymbol{dx} = \int_c^d \left[\int_a^b \boldsymbol{f(x,y)dx}\right] \boldsymbol{dy}.$$

7. Example: If $\boldsymbol{R} = \{\boldsymbol{(x,y) : 1 \le x \le 3, 0 \le y \le 2}\}$, evaluate

$$\iint_{\boldsymbol{R}} \boldsymbol{(9x^2y + 8xy^3)dA}.$$

8. If $\boldsymbol{R} = \{\boldsymbol{(x,y) : 1 \le x \le 4, 1 \le y \le 3}\}$, evaluate $\displaystyle\iint_{\boldsymbol{R}} \boldsymbol{(2x + y)dA}$.

9. Find the volume under the paraboloid $z = 1 - x^2 - y^2$ over the unit square $[0, 1] \times [0, 1]$.

10. Evaluate $\iint_R \frac{y}{1 + 2x} dA$, where $R = [0, 2] \times [0, 4]$.

11. Evaluate $\iint_R x^2 y dA$, where $R = [-1, 1] \times [0, 2]$.

12. Properties:

(a) $\iint_R cf(x, y)dA = c \iint_R f(x, y)dA$

(b) $\iint_R [f(x, y) + g(x, y)]dA = \iint_R f(x, y)dA + \iint_R g(x, y)dA$

(c) $\iint_R f(x, y)dA = \iint_{R_1} f(x, y)dA + \iint_{R_2} f(x, y)dA$, where $R = R_1 \cup R_2$.