14.8 Constrained Optimization and Lagrange Multipliers

1. Motivation:

2. Method of Lagrange Multipliers: To find the maximum and minimum values of $\boldsymbol{f(x, y, z)}$ subject to the constraint $\boldsymbol{g(x, y, z) = k}$:

   (a) Find all values of $\boldsymbol{x, y, z}$ and $\boldsymbol{\lambda}$ such that

   $$\boldsymbol{\nabla f(x, y, z) = \lambda \nabla g(x, y, z)} \quad \text{and} \quad \boldsymbol{g(x, y, z) = k}.$$

   (b) Evaluate $\boldsymbol{f}$ at all the points $\boldsymbol{(x, y, z)}$ that result from Step (a). The largest of these values is the maximum value of $\boldsymbol{f}$; the smallest is the minimum value of $\boldsymbol{f}$.

3. Example: Find the extreme values of $\boldsymbol{f(x, y) = x^2 + 2y^2}$ given the constraint $\boldsymbol{x^2 + y^2 = 1}$.

4. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

5. The production of a company is given by the Cobb-Douglas function $P = 200L^{2/3}K^{1/3}$. Cost constraints on the business force $2L + 5K \leq 150$. Find the values of the labor L and capital K to maximize production.

6. Find the maximum/minimum values of $f(x, y) = x^2 + y^2$ subject to $2x + 3y = 6$.