14.7 Optimization (Maxima and Minima)

1. Local Maximum/Minimum: A function f has a local maximum at (a, b) if
$$f(x, y) \leq f(a, b)$$
when (x, y) is near (a, b), and $f(a, b)$ is called a local maximum value. A function f has a local minimum at (a, b) if
$$f(x, y) \geq f(a, b)$$
when (x, y) is near (a, b), and $f(a, b)$ is called a local minimum value.

2. Global Maximum/Minimum: If
$$f(x, y) \leq f(a, b)$$
(or $f(x, y) \geq f(a, b)$) for all points (x, y) in the domain of f, then f has a global maximum (or global minimum) at (a, b).

3. Example: $f(x, y) = \sin x + \sin y$.

4. Example: $f(x, y) = (x^2 + y^2)e^{-x^2-y^2}$.

5. Example: $\dfrac{\cos(x^2 + y^2)}{1 + x^2 + y^2}$.

6. Fermat's Theorem: If f has a local maximum or minimum at (a, b) and the first-order partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

7. Critical Point: A point (a, b) is called a critical point (or stationary point) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

8. Example: Find the critical points and extreme values of the function
$$f(x, y) = x^2 + y^2 - 4x - 4y + 10.$$

9. Example: Find the extreme values of $f(x, y) = y^2 - x^2$.

10. Saddle Point: The point $P\big(a, b, f(a, b)\big)$ is a saddle point of $z = f(x, y)$ if (a, b) is a critical point of f and if every open disk centered at (a, b) contains points (x, y) in the domain of f for which $f(x, y) < f(a, b)$ and other points (x, y) in the domain of f for which $f(x, y) > f(a, b)$.

11. Theorem (Second Derivative Test): Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, that is (a, b) is a critical point. Let

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy})^2(a, b).$$

   (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
   (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
   (c) If $D < 0$, then $f(a, b)$ has a saddle point at (a, b).
   (d) If $D = 0$, then no conclusion (could be a local max, local min, or a saddle point at (a, b)).

12. Example: Suppose some function f has continuous second derivatives, and a critical point at $(1, 2)$. If $f_{xx}(1, 2) = 1$, $f_{xy}(1, 2) = 4$ and $f_{yy}(1, 2) = 18$, then classify the point $(1, 2)$. If instead $f_{yy}(1, 2) = k$, for what values of k is $(1, 2)$ a saddle point?

13. Find the local maximum and minimum values and all saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

14. Extreme Value Theorem: Suppose that $f(x, y)$ is continuous on the closed and bounded region $R \subset \mathbb{R}^2$. Then f has both a global maximum and a global minimum on R. Further, a global extremum may only occur at a critical point in R or at a point on the boundary of R.

15. Upshot: To find the global maximum and minimum values of a continuous function $\boldsymbol{f}$ on a closed and bounded set $\boldsymbol{D}$:

    1. Find the values of $\boldsymbol{f}$ at the critical points of $\boldsymbol{f}$ in $\boldsymbol{D}$.
    2. Find the extreme values of $\boldsymbol{f}$ on the boundary of $\boldsymbol{D}$.
    3. The largest of the values from steps 1 and 2 is the global maximum value; the smallest of these values is the global minimum value.

16. Find the global maximum and minimum values of the function $\boldsymbol{f(x,y) = x^2 - 2xy + 2y}$ on the rectangle $\boldsymbol{D = \{(x,y) :\ 0 \le x \le 3,\ 0 \le y \le 2\}}$.

17. Find the global maximum and minimum values of the function $\boldsymbol{f(x,y) = 3 + xy - x - 2y}$ on the closed triangular region with vertices $\boldsymbol{(1,0)}$, $\boldsymbol{(5,0)}$, and $\boldsymbol{(1,4)}$.

18. Examples: Find the point on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$.

Find the points on the surface $x^2y^2z = 1$ that are closest to the origin.

Find three positive numbers x, y, and z whose sum is 100 and whose product is maximum.

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $9x^2 + 36y^2 + 4z^2 = 36$.