14.5 The Gradient Vector and Directional Derivatives

1. Gradient Vector: If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

2. Find the gradient of $f(x, y) = \sin x + e^{xy}$ at $(0, 1)$.

3. Find $\nabla g(a, b)$ if $g(x, y) = xy^2 - 2xy$.

4. The air temperature at ground level (x, y) is given (in Celsius) by the function

$$T(x, y) = 20 - x - y + 0.05x^3 - 0.2y^2.$$

Find the temperature gradient at the origin and at the point $(-4, 1)$. (See picture.)

5. Recall for $z = f(x, y)$ we defined the partial derivatives at (a, b) by

6. Directional Derivative: The directional derivative of f at (a, b) in the direction of a unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is

   if this limit exists. If $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}} f(a, b) = f_x(a, b)$, and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}} f(a, b) = f_y(a, b)$.

7. Theorem: If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ and

   Using the gradient vector we may rewrite the directional derivative in the direction of the unit vector $\mathbf{u}$ as

8. Find the directional derivative of $f(x, y) = x^2y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

9. Find the directional derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

10. Find the directional derivative of $f(x, y) = x^3 - 3xy + 4y^2$ in the direction $\theta = \pi/6$, that is, in the direction that points upward and to the right at an angle of θ. What is $D_{\mathbf{u}} f(1, 2)$?

11. Chain Rule for Paths: If $f(x, y)$ is a differentiable function and $\mathbf{c}(t) = \langle x(t), y(t)\rangle$ is a differentiable path, then

$$\frac{d}{dt} f(\mathbf{c}(t)) =$$

12. Use the chain rule for paths to calculate $\frac{d}{dt} f(\mathbf{c}(t))$ for $f(x, y) = x^2 - 3xy$ and $\mathbf{c}(t) = (\cos t, \sin t)$ at $t = 0$.

13. Three Variables: The gradient for $f(x, y, z)$ is

and the directional derivative in the direction of a unit vector $\mathbf{u}$ is

14. Recall the angle between two vectors.

15. Theorem: Suppose that f is a differentiable function of two (or three) variables. Then

(a) the maximum value of the directional derivative $D_{\mathbf{u}} f(x, y)$ (the maximum rate of change of f) at (a, b) is $\|\nabla f(a, b)\|$ and it occurs when u has the same direction as the gradient vector $\nabla f(x, y)$: f increases most rapidly in the direction of the gradient;

(b) the minimum value of the directional derivative $D_{\mathbf{u}} f(x, y)$ (the minimum rate of change of f) at (a, b) is $-\|\nabla f(a, b)\|$ and it occurs when u has the opposite direction as the gradient vector $\nabla f(x, y)$: f decreases most rapidly in the direction opposite the gradient;

(c) the rate of change of f at (a, b) is 0 in the directions orthogonal to $\nabla f(a, b)$;

(d) the gradient $\nabla f(a, b)$ is orthogonal to the level curve $f(x, y) = c$ at the point (a, b), where $c = f(a, b)$.

Proof:

16. If $\boldsymbol{f(x, y) = xe^y}$, find the rate of change of $\boldsymbol{f}$ at the point $\boldsymbol{P(2, 0)}$ in the direction from $\boldsymbol{P}$ to $\boldsymbol{Q(1/2, 2)}$. In what direction does $\boldsymbol{f}$ have the maximum rate of change? What is this maximum rate of change?

17. Find the direction in which the function $\boldsymbol{f(x, y) = e^{2x+3y-1}}$ increases most rapidly, starting at the point $\boldsymbol{(3, 7)}$, and calculate its directional derivative in that direction.

18. If the elevation of a hill is given by $\boldsymbol{f(x, y) = 200 - y^2 - 4x^2}$, in which direction will rain water run off the hill at the site $\boldsymbol{(1, 2)}$?

19. Theorem: If $\boldsymbol{f(x, y, z)}$ has continuous partial derivatives at the point $\boldsymbol{(a, b, c)}$ and $\boldsymbol{\nabla f(a, b, c) \neq 0}$, then $\boldsymbol{\nabla f(a, b, c)}$ is a normal vector to the tangent plane to the surface $\boldsymbol{f(x, y, z) = k}$ at the point $\boldsymbol{(a, b, c)}$ and the equation of the tangent plane is

$$\boldsymbol{0 = f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c).}$$

Moreover the normal line through the surface at the point $\boldsymbol{(a, b, c)}$ in the direction of $\boldsymbol{\nabla f(a, b, c)}$ has parametric equations

$$\boldsymbol{x = a + f_x(a, b, c)t, \quad y = b + f_y(a, b, c)t, \quad z = c + f_z(a, b, c)t.}$$

20. Find equations of the tangent plane and normal line to $\boldsymbol{x^3y - y^2 + z^2 = 7}$ at the point $\boldsymbol{(1, 2, 3)}$.