14.3 Partial Derivatives

1. Let $f(x, y)$ be a function of two variables, where $y = b$ is fixed. Then $g(x) = f(x, b)$ is a function of a single variable x. If g has a derivative at a, then we call it the partial derivative of f with respect to x at (a, b) and write

$$f_x(a, b) = g'(a).$$

2. Now keep $x = a$ fixed, and let $h(y) = f(a, y)$. If h has a derivative at b, then we call it the partial derivative of f with respect to y at (a, b) and write

$$f_y(a, b) = h'(b).$$

3. By the definition of a derivative, we have

$$\begin{aligned} f_x(a, b) &= \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}, \\ f_y(a, b) &= \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k}. \end{aligned}$$

The partial derivatives of $f(x, y)$ are the functions $f_x(x, y)$ and $f_y(x, y)$ obtained by letting the point (a, b) vary.

4. Notation: If $z = f(x, y)$, then we may write

$$\begin{aligned} f_x(x, y) &= f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f, \\ f_y(x, y) &= f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f. \end{aligned}$$

5. To find f_x regard y as a constant and differentiate $f(x, y)$ with respect to x. To find f_y regard x as a constant and differentiate $f(x, y)$ with respect to y.

6. If $f(x, y) = x^2 + 3x^3 y - xy^2$ find $f_x(0, 1)$ and $f_y(1, 0)$.

7. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for the following functions.

$$f(x, y) = \frac{2y}{y + \cos x}$$

$$f(x, y) = e^{x^2+y^2+1}$$

$$f(x, y) = \ln(x^4 + 2y^3)$$

8. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

9. Interpretations: Partial derivative can be interpreted as rates of change. The geometric interpretation: the partial derivatives are the slopes of the tangent lines at $P(a, b, c)$ to the curves given by the intersection of the surface given by $z = f(x, y)$ and the planes $x = a$ and $y = b$.

10. If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables.

11. Second Partials: The second partial derivatives of f are

$$\begin{aligned}
(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\
(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\
(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\
(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}.
\end{aligned}$$

12. Find all second partial derivatives of

$$f(x, y) = x^3 + x^2 y^3 - 2y^2$$

13. Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

14. Calculate $\boldsymbol{f_{xxy}}$ if $\boldsymbol{f(x,y) = \sin(3x^2 + xy)}$.

15. Find the partial derivatives of $\boldsymbol{f(x,y) = \int_x^y e^{t^2+t+1} dt}$.

16. Find $\boldsymbol{f_x}$, $\boldsymbol{f_y}$, $\boldsymbol{f_{xy}}$, $\boldsymbol{f_{yx}}$ for $\boldsymbol{f(x,y) = xye^{3xy}}$.