13.4 Curvature

1. Consider a vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where t represents time. Imagine an object moving along the curve C traced out by the endpoint of this function. Recall that the arc length $s(t)$ of the curve from $u = a$ to $u = t$ is

$$s(t) = \int_a^t \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}du = \int_a^t \|\mathbf{r}'(u)\|du.$$

2. Unit Tangent Vector: Recall that $\mathbf{v}(t) = \mathbf{r}'(t)$ is both the velocity vector and a tangent vector to the curve C pointing in the direction of motion. Thus the unit tangent vector to the curve C is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

3. Find the unit tangent vector to the curve determined by

$$\mathbf{r}(t) = \langle 3t, \cos 2t, \sin 2t\rangle.$$

4. Curvature: The curvature κ (kappa) of a curve is the scalar quantity $\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\|$, the magnitude of the rate of change of the unit tangent vectors with respect to arc length along the curve. This can be rewritten as

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}, \quad \text{where} \quad \|\mathbf{r}'(t)\| \neq 0.$$

5. Find the curvature of the helix $\mathbf{r}(t) = \langle 2\cos 2t, 2\sin 2t, 3t\rangle$.

6. Curvature: The curvature κ of a curve is also given by

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\|\mathbf{a}(t) \times \mathbf{v}(t)\|}{\|\mathbf{v}(t)\|^3}.$$

Proof: Since $\mathbf{T} = \dfrac{\mathbf{r}'}{\|\mathbf{r}'\|}$ and $\|\mathbf{r}'\| = \dfrac{ds}{dt}$, we have $\mathbf{r}' = \dfrac{ds}{dt}\,\mathbf{T}$. Differentiating with respect to t results in

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\,\mathbf{T} + \frac{ds}{dt}\,\mathbf{T}'.$$

Therefore

$$\mathbf{r}' \times \mathbf{r}'' = \left(\frac{ds}{dt}\right)^2 (\mathbf{T} \times \mathbf{T}').$$

By Theorem 13 from Section 13.2, $\|\mathbf{T}(t)\| = 1$ for all t implies $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal. Thus $\theta = \frac{\pi}{2}$ is the angle between $\mathbf{T}$ and $\mathbf{T}'$, so that $\sin\theta = 1$ and

$$\|\mathbf{r}' \times \mathbf{r}''\| = \left(\frac{ds}{dt}\right)^2 \|\mathbf{T} \times \mathbf{T}'\| = \left(\frac{ds}{dt}\right)^2 \|\mathbf{T}\|\,\|\mathbf{T}'\| = \|\mathbf{r}'\|^2\,\|\mathbf{T}'\|.$$

It follows that

$$\|\mathbf{T}'\| = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^2}$$

and

$$\kappa = \frac{\|\mathbf{T}'\|}{\|\mathbf{r}'\|} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}.$$

7. Find the curvature of $\mathbf{r}(t) = \langle t^3, t^2, t\rangle$ at $t = 1$.

8. Curvature for a plane curve $y = f(x)$: For the plane curve $y = f(x)$ we have $\mathbf{r}(t) = \langle t, f(t), 0 \rangle$, and the curvature is given by

$$\kappa = \frac{|f''(x)|}{\left\{1 + \left[f'(x)\right]^2\right\}^{3/2}}.$$

9. True or False: The curvature of $y = f(x)$ is always the same as the curvature of $\mathbf{r}(t) = \langle t, f(t), c \rangle$ for any constant c.

10. Find the curvature of the circle of radius q centered at (a, b).

11. Recall the unit tangent vector to a curve is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

12. Unit Normal Vector: Since $\mathbf{T}(t)$ has constant norm (of 1), $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal. Thus the unit normal vector is a unit vector in the same direction as $\mathbf{T}'(t)$ given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

13. Osculating Circle/Circle of Curvature: A tangent line to a curve is the straight line that shares the location and direction of the curve, while an osculating circle to the same curve shares the location, direction, and curvature of the curve at that point. That is to say, if a curve has a nonzero curvature κ at point P, then the circle in the osculating plane of radius $1/\kappa$ with center lying along the normal vector $\mathbf{N}(t)$ is the osculating circle.