13.3 Arc Length and Speed

1. Arc Length in $\mathbb{R}^2$:

2. Arc Length in $\mathbb{R}^3$:

3. Arc Length in $\mathbb{R}^3$: For the curve traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where f, f', g, g', h, h' are all continuous scalar functions for $t \in [a, b]$ and the curve is traversed exactly once as t increases from a to b, the arc length s is given by

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt.$$

4. Find the arc length of the circular helix traced out by the endpoint of the vector-valued function $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.

5. Find parametric equations for the intersection of the cone $z = \sqrt{x^2 + y^2}$ and the plane $y + z = 2$ and find the length of the arc in the first octant.

6. Consider a vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, where t represents time. Imagine an object moving along the curve traced out by the endpoint of this function. Recall that $\mathbf{r}\,'(t)$ is a tangent vector pointing in the direction of the orientation of the curve.

Speed: $s'(t) = \|\mathbf{r}\,'(t)\| = v(t)$

Velocity: $\mathbf{v}(t) = \mathbf{r}\,'(t)$

Acceleration: $\mathbf{a}(t) = \mathbf{v}\,'(t) = \mathbf{r}\,''(t)$

7. Parametrizations: Compare $\mathbf{r_1}(t) = \langle \cos t, \sin t\rangle$, $\mathbf{r_2}(t) = \langle \cos \frac{t}{2}, \sin \frac{t}{2}\rangle$, and

$$\mathbf{r_3}(t) = \big\langle \cos(1+t^2), \sin(1+t^2)\big\rangle.$$

8. Arc Length Parametrization: Let $\mathbf{r}(t)$ be a parametrized curve, and set

$$s(t) = \int_a^t \|\mathbf{r}\,'(u)\| du.$$

9. Find the arc length parametrization for the helix $\mathbf{r}(t) = \langle 20t^{3/2} + 12t^{5/2}, 15t^2 + 30t \rangle$ for $t \geq 0$.

10. Find the arc length parametrization for the helix $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$.

11. Explain why the curves $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ and $\mathbf{r_1}(t) = \langle kx(t/k), ky(t/k), kz(t/k)\rangle$ have the same speed, in the sense that the speed of the moving point $\mathbf{r}$ at time t equals the speed of the moving point $\mathbf{r_1}$ at time kt.