13.2 The Calculus of Vector-Valued Functions

1. Limits: For a vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, the limit of $\mathbf{r}(t)$ as t approaches a is given by
$$\lim_{t \to a} \mathbf{r}(t) = \lim_{t \to a} \langle f(t), g(t), h(t) \rangle = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle,$$
provided all the limits exist. If any of the limits on the right-hand side above fail to exist, then $\lim_{t \to a} \mathbf{r}(t)$ does not exist.

2. Find $\displaystyle\lim_{t \to 1} \left\langle \frac{3}{t^2}, \frac{\ln t}{t^2 - 1}, \sin 2t \right\rangle$.

3. Continuity: The vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at $t = a$ whenever $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

4. Theorem on Continuity: The vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at $t = a$ if and only if the component functions f, g, h are all continuous at $t = a$.

5. Determine the values of $\mathbf{t}$ where the vector-valued function
$$\mathbf{r}(t) = \left\langle \frac{1}{t} \sin 2\pi t, \tan 2\pi t, \cos 2\pi t \right\rangle$$
is continuous.

6. Derivative: The derivative $\mathbf{r}\,'(t)$ of the vector-valued function $\mathbf{r}(t)$ is defined by
$$\mathbf{r}\,'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$
provided the limit exists. If the limit does exist, say at $t = a$, then $\mathbf{r}$ is differentiable at a.

7. Theorem on Differentiability: Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and suppose the components f, g, h are all differentiable for some value of t. Then $\mathbf{r}$ is also differentiable at t, and the derivative is given by
$$\mathbf{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle.$$

   Proof:

8. Find the derivative of $\mathbf{r}(t) = \langle e^{t^4}, \sqrt{3t^2+5}, 5/t^3 \rangle$.

9. Properties of the Derivative: Suppose $\mathbf{r}(t)$ and $\mathbf{s}(t)$ are differentiable vector-valued functions, $f(t)$ is a differentiable scalar function, and c is any scalar constant. Then

   (a) $\dfrac{d}{dt}[\mathbf{r}(t) + \mathbf{s}(t)] = \mathbf{r}\,'(t) + \mathbf{s}\,'(t)$

   (b) $\dfrac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}\,'(t)$

   (c) $\dfrac{d}{dt}[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}\,'(t) + f'(t)\mathbf{r}(t)$

   (d) $\dfrac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{s}(t)] = \mathbf{r}(t) \cdot \mathbf{s}\,'(t) + \mathbf{r}\,'(t) \cdot \mathbf{s}(t)$

   (e) $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{s}(t)] = \mathbf{r}(t) \times \mathbf{s}\,'(t) + \mathbf{r}\,'(t) \times \mathbf{s}(t)$

   (f) $\dfrac{d}{dt}\mathbf{r}(g(t)) = g'(t)\mathbf{r}\,'(g(t))$.

   Proof:

10. Tangent Vector: The tangent vector to a curve C traced out by the endpoint of the vector-valued function $\mathbf{r}(t)$ at $t = a$ is the vector $\mathbf{r}\,'(a)$.

11. Tangent Line: The tangent line to the curve $\mathbf{r}(t)$ at t_0 is parametrized as $\mathbf{L}(t) = \mathbf{r}(t_0) + t\mathbf{r}\,'(t_0)$.

12. Sketch the curve traced out by the endpoint of $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ and plot position and tangent vectors at the points where $t = 0, \pi/2, \pi$.

13. Theorem: $\|\mathbf{r}(t)\| =$ constant if and only if $\mathbf{r}(t)$ and $\mathbf{r}\,'(t)$ are orthogonal for all t.

    Proof:

14. Find the unit tangent vector for $\mathbf{r}(t) = \langle 6t^5, 4t^3, 2t \rangle$ at the point with $t = 1$.

15. Find the tangent line to the graph of $\mathbf{r}(t) = 2\cos \pi t\mathbf{i} + 2\sin \pi t\mathbf{j} + 3t\mathbf{k}$ when $t_0 = 1/3$.

16. The curves $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{s}(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at the origin. To the nearest degree, find their angle of intersection.

17. Antiderivative: The vector-valued function $\mathbf{R}(t)$ is an antiderivative of the vector-valued function $\mathbf{r}(t)$ if $\mathbf{R}\,'(t) = \mathbf{r}(t)$.

18. Indefinite Integral: If $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, the indefinite integral of $\mathbf{r}(t)$ is given by

$$\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

19. Definite Integral: For the vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, the definite integral of $\mathbf{r}(t)$ is given by

$$\int_a^b \mathbf{r}(t)dt = \int_a^b \langle f(t), g(t), h(t)\rangle dt = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle.$$

20. Fundamental Theorem of Calculus: If $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on $[a, b]$, then

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(b) - \mathbf{R}(a).$$

21. Evaluate $\displaystyle\int_0^2 \mathbf{r}(t)dt$ for $\mathbf{r}(t) = \left\langle \frac{4}{t+1}, e^{t-2}, te^t \right\rangle$.