12.4 The Cross Product

1. 2×2 Determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

2. 3×3 Determinant: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} b_1 + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} c_1.$

3. Compute $\begin{vmatrix} -2 & -5 \\ 1 & 3 \end{vmatrix}$ and $\begin{vmatrix} 2 & 3 & -1 \\ 0 & 1 & 0 \\ -2 & -1 & 3 \end{vmatrix}$.

4. Cross Product: The cross product of two vectors $\mathbf{v} = \langle a_1, b_1, c_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2, c_2 \rangle$ in $\mathbb{R}^3$ is the vector
$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \mathbf{k}.$$

5. Compute $\langle 1, -3, 5 \rangle \times \langle -2, 4, 6 \rangle$.

6. Theorem: For any vector $\mathbf{v} \in \mathbb{R}^3$ we have $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{v} \times \mathbf{0} = \mathbf{0}$.

7. Theorem: For $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ the cross product $\mathbf{v} \times \mathbf{w}$ is orthogonal (normal) to both $\mathbf{v}$ and $\mathbf{w}$.

8. Right-hand Rule: If you align the fingers of your right hand along the vector $\mathbf{v}$ and bend your fingers around in the direction of rotation from $\mathbf{v}$ toward $\mathbf{w}$ through an angle of less than $180°$, your thumb will point in the direction of $\mathbf{v} \times \mathbf{w}$. Note that $\mathbf{w} \times \mathbf{v}$ points in the opposite direction of $\mathbf{v} \times \mathbf{w}$.

9. Warning! $\mathbf{v} \times \mathbf{w} \neq \mathbf{w} \times \mathbf{v}$, and $(\mathbf{v} \times \mathbf{w}) \times \mathbf{u} \neq \mathbf{v} \times (\mathbf{w} \times \mathbf{u})$ in general.

10. Theorem: For any vectors $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^3$ and any scalar λ, the following hold:

    (a) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$ — Anticommutativity

    (b) $(\lambda \mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda \mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w})$

    (c) $\mathbf{v} \times (\mathbf{w} + \mathbf{u}) = \mathbf{v} \times \mathbf{w} + \mathbf{v} \times \mathbf{u}$ — Distributive law

    (d) $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$ — Distributive law

    (e) $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$ — Scalar triple product

    (f) $\mathbf{v} \times (\mathbf{w} \times \mathbf{u}) = (\mathbf{v} \cdot \mathbf{u})\mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$ — Vector triple product

11. Theorem: For nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, if θ is the angle between $\mathbf{v}$ and $\mathbf{w}$ and $0 \leq \theta \leq \pi$, then

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \, \sin\theta.$$

12. Corollary: Two nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are parallel if and only if $\mathbf{v} \times \mathbf{w} = \vec{0}$.

13. Find $\|\mathbf{u} \times \mathbf{v}\|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or out of the page.

14. Area of Parallelogram: The parallelogram formed by two nonparallel nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ has an area given by the norm of their cross product: Area $= \|\mathbf{v} \times \mathbf{w}\|$.

15. Compute the area of the parallelogram formed by vectors $\langle 1, 2, 3\rangle$ and $\langle 4, 5, 6\rangle$.

16. Note: $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

17. Volume of Parallelopiped and Scalar Triple Product: The parallelopiped formed by three non-coplanar vectors $\mathbf{v}, \mathbf{w}, \mathbf{u}$ has a volume given by: Volume $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$.

18. Compute the volume of the parallelopiped with three adjacent edges formed by the vectors $\mathbf{v} = \langle 1, 2, 1\rangle$, $\mathbf{w} = \langle 2, 1, -3\rangle$ and $\mathbf{u} = \langle 2, -1, 3\rangle$.

19. Torque $\vec{\boldsymbol{\tau}}$: A force known as torque $\vec{\boldsymbol{\tau}}$ is defined by $\vec{\boldsymbol{\tau}} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{F}$ is the force applied to the end of a handle and $\mathbf{r}$ is the position vector for the end of the handle. In particular,

$$\|\vec{\boldsymbol{\tau}}\| = \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\ \|\mathbf{F}\|\ \sin\theta.$$

20. Compute the magnitude of the torque if a 40-newton force is applied to the end of a 0.25-meter wrench at an angle of 75°. Give your answer in joules, where 1 joule is 1 newton-meter.

21. Point-to-Line Distance in $\mathbb{R}^3$: The distance d from a point Q to the line through the points P and R in $\mathbb{R}^3$ is

$$d =$$

22. Compute the distance from $Q(1, 2, 1)$ to the line through the points $P(2, 1, -3)$ and $R(2, -1, 3)$.