12.3 The Dot Product

1. Dot Product: The dot product of two vectors $\mathbf{v} = \langle a_1, b_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2 \rangle$ in $\mathbb{R}^2$ is the scalar

   $\mathbf{v} \cdot \mathbf{w} =$

   Similarly, the dot product of two vectors $\mathbf{v} = \langle a_1, b_1, c_1 \rangle$ and $\mathbf{w} = \langle a_2, b_2, c_2 \rangle$ in $\mathbb{R}^3$ is

   $\mathbf{v} \cdot \mathbf{w} =$

2. Compute the dot product $\mathbf{v} \cdot \mathbf{w}$ for the following:

   (a) $\mathbf{v} = \langle 3, -2, -4 \rangle$ and $\mathbf{w} = \langle 2, 1, 5 \rangle$;

   (b) $\mathbf{v} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

3. Theorem: For vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{u}$ and any scalar λ the following hold:

   (a) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ — Commutative law

   (b) $\mathbf{v} \cdot (\mathbf{w} + \mathbf{u}) = \mathbf{v} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{u}$ — Distributive law

   (c) $(\lambda \mathbf{v}) \cdot \mathbf{w} = \lambda(\mathbf{v} \cdot \mathbf{w}) = \mathbf{v} \cdot (\lambda \mathbf{w})$

   (d) $\mathbf{0} \cdot \mathbf{w} = 0$

   (e) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.

   Proof:

4. Angle between vectors: For two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$, define the angle θ, $0 \le \theta \le \pi$, to be the smaller angle between the vectors.

5. Theorem: Let θ be the angle between nonzero vectors $\mathbf{v}$ and $\mathbf{w}$. Then

   $$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \, \|\mathbf{w}\| \, \cos\theta.$$

   Proof:

6. Compute the angle between the vectors $\mathbf{v} = \langle 2, 3, 5 \rangle$ and $\mathbf{w} = \langle -4, 1, -1 \rangle$.

7. Orthogonal: Two vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal iff $\mathbf{v} \cdot \mathbf{w} = 0$.

8. Find all values of λ such that $\mathbf{v} = \langle \lambda, -2, 3 \rangle$ and $\mathbf{w} = \langle \lambda, \lambda, -5 \rangle$ are orthogonal.

9. The angle between $\mathbf{v}$ and $\mathbf{w}$ is acute if $\mathbf{v} \cdot \mathbf{w} > 0$ and obtuse if $\mathbf{v} \cdot \mathbf{w} < 0$.

10. Find a vector orthogonal to $4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

11. Component of $\mathbf{v}$ along $\mathbf{w}$: The component of $\mathbf{v}$ along $\mathbf{w}$ is the scalar

$$\mathrm{comp}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|} = \mathbf{v} \cdot \mathbf{e}_{\mathbf{w}} = \|\mathbf{v}\| \cos \theta.$$

12. Projection of $\mathbf{v}$ onto $\mathbf{w}$: The projection of $\mathbf{v}$ onto $\mathbf{w}$ is the vector

$$\mathbf{v}_{\|} = \mathrm{proj}_{\mathbf{w}}\mathbf{v} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \right) \mathbf{w} = (\mathbf{v} \cdot \mathbf{e}_{\mathbf{w}})\, \mathbf{e}_{\mathbf{w}} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \right) \mathbf{w}.$$

13. Compute $\mathrm{comp}_{\mathbf{w}}\mathbf{v}$ and $\mathrm{proj}_{\mathbf{w}}\mathbf{v}$ for $\mathbf{v} = \langle 2, -1, 3 \rangle$ and $\mathbf{w} = \langle 1, 2, 2 \rangle$.

14. Prove that $\mathbf{v} \cdot \mathbf{w} = \frac{1}{4}\|\mathbf{v} + \mathbf{w}\|^2 - \frac{1}{4}\|\mathbf{v} - \mathbf{w}\|^2$.