12.1 Vectors in the Plane

1. A vector is any quantity with a length (magnitude, norm) and a direction. We typically draw an arrow to represent a vector. For example, let $\mathbf{v}$ be the vector from initial point A to terminal point B. Then $\mathbf{v} = \overrightarrow{AB}$, and the magnitude is denoted $\|\mathbf{v}\| = \|\overrightarrow{AB}\|$.

2. Since the velocity of an object in motion is the speed of the object in a particular direction, velocity is an example of a vector, where speed=magnitude in this case.

   We could also interpret $\overrightarrow{AB}$ as representing the path of a moving object, moving from initial point A to terminal point B, with $\|\overrightarrow{AB}\|$ representing the total distance.

3. Equivalent vectors: same magnitude, same direction.

4. Vector addition

5. Scalar multiplication

6. A position vector begins at the origin O and terminates at some point $P(a, b)$. Let $\mathbf{v} = \overrightarrow{OP} = \langle a, b \rangle$. We call a and b the components of $\mathbf{v}$. Clearly the magnitude or norm of $\mathbf{v}$ is

   $$\|\mathbf{v}\| = \sqrt{a^2 + b^2}$$

   by the Pythagorean theorem.

7. For two position vectors $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$ we have $\mathbf{v} = \mathbf{w}$ if and only if $v_1 = w_1$ and $v_2 = w_2$. Componentwise equality.

8. Finding a position vector: Find the vector with

   (a) initial point $A(-3, 2)$ and terminal point $B(1, 4)$;

   (b) initial point $A(v_1, v_2)$ and terminal point $B(w_1, w_2)$.

   (c) Find $\|\overrightarrow{AB}\|$.

9. Vector addition and subtraction: For two position vectors $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$ we have

   $$\mathbf{v} + \mathbf{w} = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle = \langle v_1 + w_1, v_2 + w_2 \rangle.$$

   Componentwise addition. Similarly, subtraction is given by

   $$\mathbf{v} - \mathbf{w} = \langle v_1, v_2 \rangle - \langle w_1, w_2 \rangle = \langle v_1 - w_1, v_2 - w_2 \rangle.$$

10. Scalar multiplication: For scalar λ and position vector $\mathbf{v} = \langle v_1, v_2 \rangle$, we have $\lambda \mathbf{v} = \lambda \langle v_1, v_2 \rangle = \langle \lambda v_1, \lambda v_2 \rangle$. Note that $\|\lambda \mathbf{v}\| = |\lambda| \, \|\mathbf{v}\|$.

11. Parallel: Two vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if there exists a scalar $\lambda \neq 0$ such that $\mathbf{w} = \lambda \mathbf{v}$. Thus two vectors are parallel if they point in the same or opposite direction.

12. For vectors $\mathbf{v} = \langle 4, -2 \rangle$ and $\mathbf{w} = \langle -3, 5 \rangle$, compute

    (a) $\mathbf{v} + \mathbf{w}$

    (b) $\frac{1}{2}\mathbf{v} + 10\mathbf{w}$

    (c) $5\mathbf{v} - 3\mathbf{w}$

    (d) $\| - 2\mathbf{w}\|$

13. Are $\mathbf{v} = \langle -4, 6 \rangle$ and $\mathbf{w} = \langle 3, -5 \rangle$ parallel?

    Pick x so that $\mathbf{v}$ and $\mathbf{u} = \langle 3, x \rangle$ are parallel.

14. Standard basis vectors: For any position vector $\mathbf{v} = \langle v_1, v_2 \rangle$ we have

$$\mathbf{v} = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle.$$

Define the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$ by

$$\mathbf{i} = \langle 1, 0 \rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1 \rangle.$$

Then $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j}$.

15. Algebraic Properties: For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and real scalars c and d:

   (a) $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ — Commutativity

   (b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ — Associativity

   (c) $\mathbf{v} + \mathbf{0} = \mathbf{v}$ — Additive Identity

   (d) $\mathbf{v} + (-\mathbf{v}) = \mathbf{v} - \mathbf{v} = \mathbf{0}$ — Additive Inverse

   (e) $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$ — Distributive law

   (f) $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$ — Distributive law

   (g) $(1)\mathbf{v} = \mathbf{v}$ — Multiplication by 1

   (h) $(0)\mathbf{v} = \mathbf{0}$ — Multiplication by 0

16. Unit vectors: A vector with length 1 is called a unit vector. For any nonzero position vector $\mathbf{v} = \langle v_1, v_2 \rangle$, a unit vector pointing in the same direction as $\mathbf{v}$ is

$$\mathbf{e_v} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

17. Find a unit vector in the same direction as $\mathbf{v} = \langle -12, 5 \rangle$.

18. Linear combination: $c\mathbf{v} + d\mathbf{w}$

    If $\mathbf{v} = 5\mathbf{i} + \mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - 7\mathbf{j}$, express $3\mathbf{v} - 2\mathbf{w}$ as a linear combination of $\mathbf{i}$ and $\mathbf{j}$.

19. An airplane has an airspeed of 650 kilometers per hour (kph). Suppose the wind velocity is given by the vector $\mathbf{w} = \langle 32, 48 \rangle$. In what direction should the airplane head in order to fly due west, in other words in the direction of $-\mathbf{i}$?

20. Two forces $\mathbf{F_1}$ and $\mathbf{F_2}$ with magnitudes 10 lbs and 12 lbs, respectively, act on an object at a point P. Find the resultant force $\mathbf{F}$, its magnitude, and its angle. (See diagram)