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Stability and bifurcation analysis of a prey–predator model with age based predation

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ABSTRACT

It is observed that in large animals only adult predators take part in direct predation while suckling feed on milk of adult predators and juveniles are dependent on the dead prey stock killed by the adult predators. Some parts of the dead prey population is consumed by adult predators and remaining parts are consumed by juveniles and the remaining portion decays naturally. In light of this, a mathematical model is proposed to study the stability and bifurcation behaviour of a prey–predator system with age based predation. All the feasible equilibria of the system are obtained and the conditions for the existence of the interior equilibrium are determined. The local stability analysis of all the feasible equilibria is carried out and the possibility of Hopf-bifurcation of the interior equilibrium is studied. Finally, numerical simulation is conducted to support the analytical results.

1. Introduction

The prey predator system is a very important system in ecology which has been studied by many mathematicians [1–7]. In nature, the prey–predator system exhibits age and stage based dynamics that has been widely studied using mathematical models [8]. Recently, autonomous systems with a stage structure have been considered in [9–12]. A predator–prey system with stage structure for the prey is studied by Cui and Takeuchi [13]. They provided a sufficient and necessary condition to guarantee permanence of the system. Stability and Hopf-bifurcation analysis in a prey–predator system with stage structure for prey and time delay is studied by Chen and Changming [14]. A robust prey-dependent consumption predator prey Gomportz model with periodic harvesting for the prey and stage structure for the predator with constant maturation time delay has been studied by Liu et al. [3]. A three species Lotka–Volterra type food chain model with stage structure and time delays is investigated by Xu et al. [15]. They assumed that the individuals in each species may belong to immature or mature class. The age to maturity was presented by time delay. A ratio dependent predator–prey model with stage structure for the predator and time delay due to gestation of the predator is investigated by Xu and Ma [16]. Recently, many authors studied different kinds of stage structured models and some significant work have been carried out by Sun et al. [17–19].

Keeping this in mind, in the present paper, we have constructed a model in which age of the predator is considered to play an important role in community dynamics and rate of predation. Since most of the predators in forest in higher age-groups are large in size so it becomes essential to consider age-structure in predator population. In large animals only adult predators take part in direct predation while suckling feed on milk of adult predators and juveniles are dependent on the prey population killed by the adult predators. Some part of the dead prey-population is consumed by adult predators and some part by juveniles while the remaining part decays naturally. Therefore, in this paper a mathematical model has been

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developed by taking age structure in the predator population with Holling type-II functional response. Stability and bifurcation analysis is carried out for feasible equilibrium points.

2. Basic assumptions and mathematical model

The model is given by following system of non-linear ordinary differential equations (Table 1):

$$\frac{dP}{dT} = \hat{r}P\left(1 - \frac{P}{\widehat{K}}\right) - \frac{\hat{c}Q_3P}{(\hat{a}+P)}, \tag{1}$$

$$\frac{dP_e}{dT} = \frac{\hat{c}Q_3P}{(\hat{a}+P)} - \hat{\alpha}_1P_eQ_3 - \hat{\alpha}_2P_eQ_2 - \hat{g}P_e, \tag{2}$$

$$\frac{dQ_1}{dT} = \hat{h_1}Q_3 - (\hat{m}_1 + \hat{d}_1)Q_1, \tag{3}$$

$$\frac{dQ_2}{dT} = \hat{m}_1Q_1 + \hat{b_1}\hat{\alpha}_2P_eQ_2 - (\hat{m}_2 + \hat{d}_2)Q_2, \tag{4}$$

$$\frac{dQ_3}{dT} = \hat{m}_2Q_2 + \hat{b_2}\hat{\alpha}_1P_eQ_3 - \hat{d}_3Q_3, \tag{5}$$

with non-negative initial conditions $P(0) > 0,\ P_e = 0,\ Q_i(0) > 0, \quad i = 1,2,3.$

The above system of equations can be non-dimensionalised using the relations: $x_1 = \frac{P}{\hat{a}},\ x_2 = \frac{P_e}{\hat{a}}, x_3 = \frac{Q_1}{\hat{a}},\ x_4 = \frac{\hat{\alpha}_2Q_2}{\hat{r}},\ x_5 = \frac{\hat{c}Q_3}{\hat{r}\hat{a}},\ t = \hat{r}T$ and introducing the new parameters as $K = \frac{\widehat{K}}{\hat{a}},\ a = \frac{\hat{\alpha}_1\hat{a}}{\hat{c}},\ c = \frac{\hat{g}}{\hat{r}},\ r = \frac{\hat{h_1}}{\hat{c}},\ d_1 = \frac{(\hat{m}_1+\hat{d}_1)}{\hat{r}},\ d_2 = \frac{(\hat{m}_2+\hat{d}_2)}{\hat{r}},\ m_1 = \frac{\hat{m}_1\hat{a}\hat{\alpha}_2}{\hat{r}^2},\ m_2 = \frac{\hat{m}_2\hat{a}}{\hat{r}\hat{c}},\ d_3 = \frac{\hat{d}_3}{\hat{r}},\ b = \frac{\hat{b_1}\hat{\alpha}_2\hat{a}}{\hat{r}},\ e = \frac{\hat{b_2}\hat{\alpha}_1\hat{a}}{\hat{r}}$. The non-dimensionalised system of equations are as follows:

$$\frac{dx_1}{dt} = x_1\left(1 - \frac{x_1}{K}\right) - \frac{x_1x_5}{(1+x_1)}, \tag{6}$$

$$\frac{dx_2}{dt} = \frac{x_1x_5}{(1+x_1)} - ax_2x_5 - x_2x_4 - cx_2, \tag{7}$$

$$\frac{dx_3}{dt} = rx_5 - d_1x_3, \tag{8}$$

$$\frac{dx_4}{dt} = m_1x_3 + bx_2x_4 - d_2x_4, \tag{9}$$

$$\frac{dx_5}{dt} = m_2x_4 + ex_2x_5 - d_3x_5, \tag{10}$$

with initial conditions: $x_i(0) = x_{i0} > 0$, and $x_2(0) = 0\ i = 1,3,4,5$ where, $r,\ K,\ m_1, m_2,\ d_1,\ d_2,\ d_3,\ a,\ b,\ c$ and e are positive constants.

3. Boundedness and equilibria of the system

In this section, we analyse the system of Eqs. (6)–(10) under the initial condition $x_1(0) = x_{10} > 0,\ x_2(0) = 0,\ x_3(0) = x_{30} > 0,\ x_4(0) = x_{40} > 0,\ x_5(0) = x_{50} > 0$. The right hand side of the Eqs. (6)–(10) are smooth functions of variables $(x_1,\ x_2,\ x_3,\ x_4,\ x_5)$ and all the parameters involved in the system are non-negative. In the following lemma we have shown that all the solutions are bounded in the region Ω.

Table 1
The definition of parameters used in system of Eqs. (1)–(5).

P	Density of prey
P_e	Concentration/amount of dead prey
Q_1	Density of the suckling in predator population
Q_2	Density of the juveniles in predator population
Q_3	Density of the adults in predator population
$\hat{b}_i$	Biomass conservation rate constant
$\hat{\alpha}_i$	Consumption rates
$\hat{m}_i$	Maturation rate
$\hat{r}$	intrinsic growth rate of prey
$\hat{d}_i$	death rate
$\widehat{K}$	carrying capacity
$\hat{h_1}$	Birth rate of predator
$\hat{a}$	Half Saturation Constant
$\hat{g}$	Natural decay rate
$\hat{c}$	Predation rate

The indices i may take on the integer value 1, 2 or 3.

Lemma 3.1. *The system (6)–(10) is uniformly bounded in* Ω, *where*

$$\Omega = \{(x_1, x_2, x_3, x_4, x_5) : 0 \leqslant x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t) \leqslant \frac{K\theta}{\theta_1}\}, \text{ and}$$

$$\theta_1 = min\{(\theta - 1), c, (d_1 - m_1), (d_2 - m_2), (d_3 - r)\}.$$

Proof. We have from (6) $\frac{dx_1}{dt} \leqslant x_1(1 - \frac{x_1}{K})$. Hence $\lim_{t\to\infty} x_1 \leqslant K$. Let us consider a time dependent function: $W_1(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t)$. Clearly,

$$\frac{dW_1}{dt} = \frac{dx_1}{dt} + \frac{dx_2}{dt} + \frac{dx_3}{dt} + \frac{dx_4}{dt} + \frac{dx_5}{dt}.$$

Using (6)–(10) in the above expression, we obtain

$$\begin{aligned}\frac{dW_1}{dt} &= \left(x_1 - \frac{x_1^2}{K} - ax_2x_5 - x_2x_4 - cx_2 + rx_5 - d_1x_3 + m_1x_3 + bx_2x_4 - d_2x_4 + m_2x_4 - d_3x_5 + ex_2x_5\right) \\ &\leqslant (x_1 - cx_2 - (d_1 - m_1)x_3 - (d_2 - m_2)x_4 - (d_3 - r)x_5) \\ &\leqslant (\theta x_1 - (\theta - 1)x_1 - cx_2 - (d_1 - m_1)x_3 - (d_2 - m_2)x_4 - (d_3 - r)x_5) \leqslant \theta K - \theta_1 W_1(t),\end{aligned}$$

where θ_1 is chosen as the minimum of $\{(\theta - 1),\ c,\ (d_1 - m_1),\ (d_2 - m_2),\ (d_3 - r)\}$. Thus

$$\frac{dW_1}{dt} + \theta_1 W_1 \leqslant \theta K.$$

Now applying the theorem of differential inequalities [20], we obtain

$$0 < W_1(t) \leqslant W_1(0)e^{-\theta_1 t} + \frac{\theta K}{\theta_1},$$

as $t \to \infty$, we have

$$0 \leqslant W_1 \leqslant \frac{\theta K}{\theta_1}.$$

Hence all the solutions of the system (6)–(10) are bounded in Ω.

We now find all the possible equilibria of the system (6)–(10). The system of Eqs. (6)–(10) have three feasible equilibria, namely,

(i) Trivial equilibrium point: $E_T \equiv (0, 0, 0, 0, 0)$.
(ii) Axial equilibrium: $E_A \equiv (K, 0, 0, 0, 0)$.
(iii) Positive interior equilibrium: $E^* \equiv (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$, where $x_3^* = \frac{rx_5^*}{d_1},\ x_4^* = \alpha x_5^*,\ x_2^* = \frac{d_3 - m_2\alpha}{e},\ x_5^* = \left(1 - \frac{x_1^*}{K}\right)(1 + x_1^*)$ and x_1^* is given by $x_1^{*2}\tilde{l} + x_1^*((K-1)(1-\tilde{l}) - K\left(1 - \tilde{l} - cx_2^*\right)) = 0$, where $\tilde{l} = (ax_2^*(1-\alpha)),\ \alpha = \frac{d_3b - d_2e + \sqrt{(d_3b - d_2e)^2 + \frac{4m_2bem_1r}{d_1}}}{2m_2b}$ is exist if (i) $\alpha < 1$, (ii) $\tilde{l} < 1$, (iii) $x_2^* < \frac{1-\tilde{l}}{c}$, $x_1^* < K$ and $K > 1$. □

4. Dynamic behaviour and Hopf-bifurcation

In the Section 3, we observed that the system of Eqs. (6)–(10) have three equilibria, namely, $E_T(0, 0, 0, 0, 0)$, $E_A(K, 0, 0, 0, 0)$ and $E^*(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*)$. We will now study the dynamical behaviour of the system about all the three feasible equilibria. The variational matrix for the system of Eqs. (6)–(10) is

$$V_1 = \begin{bmatrix} 1 - \frac{2x_1}{K} - \frac{x_5}{(1+x_1)^2} & 0 & 0 & 0 & -\frac{x_1}{(1+x_1)} \\ \frac{x_5}{(1+x_1)^2} & -(ax_5 + x_4 + c) & 0 & -x_2 & -ax_2 + \frac{x_1}{(1+x_1)} \\ 0 & 0 & -d_1 & 0 & r \\ 0 & bx_4 & m_1 & -d_2 + bx_2 & 0 \\ 0 & ex_5 & 0 & m_2 & -d_3 + ex_2 \end{bmatrix}.$$

The characteristic equation of V_1 at E_T is

$$(-1+\lambda)(c+\lambda)(\lambda^3+B_1\lambda^2+B_2\lambda+B_3)=0,$$

where $B_1=d_1+d_2+d_3,\ B_2=d_1d_2+d_1d_3+d_2d_3$ and $B_3=d_1d_2d_3-rm_1m_2$.

The above characteristic equation has at least one positive root and therefore the equilibrium point E_T is unstable.

The characteristic equation of V_1 at E_A is

$$(1+\lambda)(c+\lambda)(\lambda^3+B_1\lambda^2+B_2\lambda+B_3)=0,$$

all the eigenvalue of the above equation has negative real part if $d_1d_2d_3>rm_1m_2$. So the equilibrium point E_A is asymptotically stable.

The characteristic equation of V_1 at E^* is

$$\lambda^5+A_1\lambda^4+A_2\lambda^3+A_3\lambda^2+A_4\lambda+A_5=0, \tag{11}$$

where

$$A_1=a_{11}+a_{22}+a_{44}+a_{55}+d_1,$$

$$A_2=a_{11}a_{22}+a_{11}a_{44}+a_{22}a_{44}+a_{25}a_{52}+a_{11}a_{55}+a_{22}a_{55}+a_{44}a_{55}+a_{11}d_1+a_{22}d_1+a_{44}d_1+a_{55}d_1+a_{42}x_2^*,$$

$$\begin{aligned}A_3={}&a_{15}a_{21}a_{52}+a_{25}a_{44}a_{52}+a_{22}a_{44}a_{55}+a_{22}a_{44}d_1+a_{25}a_{52}d_1+a_{22}a_{55}d_1+a_{44}a_{55}d_1+a_{25}a_{42}m_2-m_1m_2r+a_{42}a_{55}x_2^*\\&+a_{42}d_1x_2^*+a_{11}\left(a_{25}a_{52}+a_{44}a_{55}+a_{44}d_1+a_{55}d_1+a_{22}(a_{44}+a_{55}+d_1)+a_{42}x_2^*\right),\end{aligned}$$

$$\begin{aligned}A_4={}&a_{25}a_{44}a_{52}d_1+a_{22}a_{44}a_{55}d_1+a_{25}a_{42}d_1m_2+a_{15}a_{21}(a_{44}a_{52}+a_{52}d_1+a_{42}m_2)-a_{22}m_1m_2r+a_{42}a_{55}d_1x_2^*+a_{52}m_1rx_2^*\\&+a_{11}\left(a_{25}a_{44}a_{52}+a_{22}a_{44}a_{55}+a_{22}a_{44}d_1+a_{25}a_{52}d_1+a_{22}a_{55}d_1+a_{44}a_{55}d_1+a_{25}a_{42}m_2-m_1m_2r+a_{42}a_{55}x_2^*+a_{42}d_1x_2^*\right),\end{aligned}$$

$$A_5=a_{15}a_{21}d_1(a_{44}a_{52}+a_{42}m_2)+a_{11}\left(a_{22}a_{44}a_{55}d_1+a_{25}d_1(a_{44}a_{52}+a_{42}m_2)-a_{22}m_1m_2r+a_{42}a_{55}d_1x_2^*+a_{52}m_1rx_2^*\right),$$

$a_{11}=\frac{-x_1^*x_5^*}{(1+x_1^*)^2}+\frac{x_1^*}{K},\ a_{15}=\frac{x_1^*}{(1+x_1^*)},\quad a_{21}=\frac{x_5^*}{(1+x_1^*)^2},\ a_{22}=ax_5^*+x_4^*+c,\ a_{25}=ax_2^*-\frac{x_1^*}{(1+x_1^*)},\quad a_{42}=bx_4^*,\ a_{52}=ex_5^*,\ a_{44}=d_2-bx_4^*,\ a_{55}=d_3-ex_2^*$. The Routh–Hurwitz criterion gives a set of necessary and sufficient conditions for all the roots of the Eq. (11) to have negative real part and which are as follows: $A_i>0,\ i=1,2,3,4,5,\ A_1A_2>A_3,\ A_1A_2A_3>\left(A_3^2+A_1^2A_4\right)$ and $(A_3A_4-A_2A_5)(A_1A_2-A_3)>(A_1A_4-A_5)^2$. From these expressions it is however difficult to interpret the results in ecological terms.

If one of the above mentioned conditions is violated then the system would become unstable around the positive interior equilibrium point E^*.

Now, we will study the Hopf-bifurcation [21] of the above system, taking r as the bifurcation parameter. Now, the necessary and sufficient condition for the existence of the Hopf-bifurcation, if it exists is $r=r_0$ such that

(i) $A_i(r_0)>0,\ i=1,2,3,4,5,$
(ii) $A_1(r_0)A_2(r_0)>A_3(r_0),$
(iii) $A_1(r_0)A_2(r_0)A_3(r_0)>(A_3(r_0)^2+A_1(r_0)^2A_4(r_0)),$
(iV) $(A_3(r_0)A_4(r_0)-A_2(r_0)A_5(r_0))(A_1(r_0)A_2(r_0)-A_3(r_0))-(A_1(r_0)A_4(r_0)-A_5(r_0))^2=0$ and
(V) if we consider the eigen values of the characteristic Eq. (11) is of the form $\lambda_i=u_i+i\upsilon_i$, then $\frac{du_i}{dr}\neq 0,\ i=1,2,3,4,5.$

After substituting the values, the condition $(A_3A_4-A_2A_5)(A_1A_2-A_3)-(A_1A_4-A_5)^2$ becomes

$$D_1r^3+D_2r^2+D_3r+D_4=0, \tag{12}$$

where $D_1=n_2^2n_4,\ D_2=-n_2^2n_3-(-A_1n_4+n_6)^2+n_2(A_1A_2n_4-2n_1n_4+A_2n_6),\ D_3=2n_1n_2n_3+n_1^2n_4+2A_1^2n_3n_4-A_2n_2n_5-A_2n_1n_6+2n_5n_6+A_1(-A_2(n_2n_3+n_1n_4)+A_2^2n_6-2(n_4n_5+n_3n_6)),\ D_4=A_1A_2n_1n_3-n_1^2n_3-A_1^2n_3^2-A_1A_2^2n_5+A_2n_1n_5+2A_1n_3n_5-n_5^2,\ n_1=A_3+n_2r,\quad n_2=m_1m_2,\ n_3=A_4+n_4r,\ n_4=a_{22}m_1m_2-a_{52}m_1x_2^*+a_{11}m_1m_2,\ n_5=A_5+n_6r,\ n_6=a_{11}m_1(a_{22}m_2-a_{52}m_1x_2^*)$.The Eq.(12) has at least one positive root say $r=r_0$.

Therefore, one pair of eigenvalues of the characteristic Eq.(11) at $r=r_0$ are of the form $\lambda_{1,2}=\pm i\upsilon$, where υ is positive real number.

Now, we will verify the Hopf-bifurcation condition (v), putting $\lambda=u+i\upsilon$ in (11) and separating real and imaginary parts, we have

$$u^5+A_1u^4+(A_2-10\upsilon^2)u^3+(A_3-6A_1\upsilon^2)u^2+(5\upsilon^4-3A_2\upsilon^2+A_4)u+(A_1\upsilon^4-A_3\upsilon^2+A_5)=0, \tag{13}$$

$$(\upsilon^2)^2-(10u^2+4A_1u+A_2)\upsilon^2+(5u^4+4A_1u^3+3A_2u^2+2A_3u+A_4)=0. \tag{14}$$

Substituting the value of υ^2 from (14) in (13), we get

$$u^5+A_1u^4+(A_2-10f(u))u^3+(A_3-6A_1f(u))u^2+(5(f(u))^2-3A_2f(u)+A_4)u+(A_1(f(u))^2-A_3f(u)+A_5)=0,$$

where $f(u)=\frac{1}{2}((10u^2+4A_1u+A_2)-F)$, and $F=\sqrt{(10u^2+4A_1u+A_2)^2-4(5u^4+4A_1u^3+3A_2u^2+2A_3u+A_4)}$, differentiating with respect to r and putting $r=r_0$, we have

$$\left[\frac{du}{dr}\right]_{r=r_0}=\frac{f(0)\frac{dA_3}{dr_0}-(f(0))^2\frac{dA_1}{dr_0}-\frac{dA_5}{dr_0}}{5(f(0))^2+2A_1f(0)f'(0)-3A_2f(0)-A_3f'(0)+A_4}\neq 0,$$

since,

$$f(u)\frac{dA_3}{dr_0}-(f(u))^2\frac{dA_1}{dr_0}-\frac{dA_5}{dr_0}=m_1\left(a_{11}(a_{22}m_2-a_{52}x_2^*)-f(u)m_2\right)\neq 0.$$

This ensures that the above system has a Hopf-bifurcation around the interior equilibrium E^*. Hence as the ratio of growth rate of suckling to predation rate (r) crosses its threshold value, $r=r_0$, then all population starts oscillating around the interior equilibrium point.

This ensures that the above system has a Hopf-bifurcation around the interior equilibrium E^*. Now, we further reduce the set of differential Eqs. (6)–(10) into the normal form in order to determine the direction and stability criterion of the bifurcating periodic solution. Here, the Poincare's method is used to put Eqs. (6)–(10) into the normal form following the procedure outlined by Hassard et al. [22]. For the sake of simplicity, introducing the new variables $x_i=x_i^*+w_i,\ i=1,2,3,4,5$, the system of Eqs. (6)–(10) can be written in matrix form as

$$\dot{X}=AX+B, \tag{15}$$

where dot (.) cover X denotes the derivative with respect to time. Here AX is the linear part of the system and B represents the nonlinear part. Moreover,

$$X=\begin{pmatrix} w_1\\ w_2\\ w_3\\ w_4\\ w_5\end{pmatrix},\quad A=\begin{pmatrix} a_{11} & 0 & 0 & 0 & -a_{15}\\ a_{21} & -a_{22} & 0 & -x_2^* & a_{25}\\ 0 & 0 & -d_1 & 0 & r\\ 0 & a_{42} & m_1 & -a_{44} & 0\\ 0 & a_{52} & 0 & m_2 & -a_{55}\end{pmatrix},\quad B=\begin{pmatrix} b_{11}\\ b_{12}\\ 0\\ bw_2w_4\\ ew_2w_5\end{pmatrix},$$

where $b_{11}=H^3x_5^*w_1^2-\frac{w_1^2}{K}-x_5^*H^4w_1^3-H^2w_1w_5+H^3w_1^2w_5$, $b_{12}=-b_{11}-\frac{w_1^2}{K}-aw_2w_5-w_2w_4$ and $H=\frac{1}{(1+x_1^*)}$. At $r=r_0$, using the equation $(A_3A_4-A_2A_5)(A_1A_2-A_3)-(A_1A_4-A_5)^2=0$ in (11) and becomes

$$(\lambda^2+\upsilon^2)\left(\lambda^3+A_1\lambda^2+\frac{1}{2}\left(A_2+\sqrt{A_2^2-4A_4}\right)\lambda+(A_3-A_1\upsilon^2)\right)=0,$$

where $\upsilon^2=\frac{1}{2}\left(A_2-\sqrt{A_2^2-4A_4}\right)$. From the above equation it is clear that $\lambda_{1,2}=\pm i\upsilon$ and other eigenvalues have negative real numbers say $-p_j,\ j=1-3$. Next, we seek a transformation matrix P which reduces the matrix A to the form

$$P^{-1}AP=\begin{pmatrix} 0 & -\upsilon & 0 & 0 & 0\\ \upsilon & 0 & 0 & 0 & 0\\ 0 & 0 & -p_1 & 0 & 0\\ 0 & 0 & 0 & -p_2 & 0\\ 0 & 0 & 0 & 0 & -p_3\end{pmatrix},$$

where the nonsingular matrix P is given as

$$P==\begin{pmatrix} 1 & 0 & 1 & 1 & 1\\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25}\\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35}\\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45}\\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55}\end{pmatrix},$$

where,

$$c_{21}=\frac{l_2(-c_{52}l_3+c_{51}l_4)+l_1(c_{51}l_3+c_{52}l_4)}{l_1^2+l_2^2},\quad c_{31}=-\frac{r(v^2+a_{11}d_1)}{a_{15}(v^2+d_1^2)},\quad c_{41}=\frac{l_6l_2+l_1l_5}{l_1^2+l_2^2},\quad c_{51}=\frac{-a_{11}}{a_{15}},$$
$$c_{22}=\frac{-l_1(-c_{52}l_3+c_{51}l_4)+l_2(c_{51}l_3+c_{52}l_4)}{l_1^2+l_2^2},\quad c_{32}=\frac{rv(-a_{11}+d_1)}{a_{15}(v^2+d_1^2)},\quad c_{42}=\frac{-l_6l_1+l_2l_5}{l_1^2+l_2^2},\quad c_{52}=\frac{v}{a_{15}},$$
$$c_{23}=\frac{-c_{33}m_1m_2+c_{53}(a_{44}-p_1)(a_{55}-p_1)}{a_{42}m_2+a_{52}(a_{44}-p_1)},\quad c_{33}=\frac{rc_{53}}{d_1-p_1},\quad c_{43}=\frac{m_1c_{33}a_{52}+a_{42}c_{53}(a_{55}-p_1)}{a_{52}(a_{44}-p_1)+a_{42}m_2},$$
$$c_{53}=\frac{-(a_{11}+p_1)}{a_{15}},\quad c_{24}=\frac{-c_{34}m_1m_2+c_{54}(a_{44}-p_2)(a_{55}-p_2)}{a_{42}m_2+a_{52}(a_{44}-p_2)},\quad c_{34}=\frac{rc_{54}}{d_1-p_2},\quad c_{35}=\frac{rc_{55}}{d_1-p_3},$$
$$c_{44}=\frac{m_1c_{34}a_{52}+a_{42}c_{54}(a_{55}-p_2)}{a_{52}(a_{44}-p_2)+a_{42}m_2},\quad c_{54}=\frac{-(a_{11}+p_2)}{a_{15}},\quad c_{25}=\frac{-c_{35}m_1m_2+c_{55}(a_{44}-p_3)(a_{55}-p_3)}{a_{42}m_2+a_{52}(a_{44}-p_3)},$$
$$c_{45}=\frac{m_1c_{35}a_{52}+a_{42}c_{55}(a_{55}-p_3)}{a_{52}(a_{44}-p_3)+a_{42}m_2},\quad c_{55}=\frac{-(a_{11}+p_3)}{a_{15}},$$
$$l_1=a_{44}a_{52}+a_{42}m_2,\quad l_2=va_{52},\quad l_3=-v^2+a_{44}a_{55}-c_{31}m_1m_2,$$
$$l_4=v(a_{44}+a_{55})+c_{32}m_1m_2,\quad l_5=a_{42}(a_{55}c_{51}+vc_{52})+a_{52}c_{31}m_1,$$
$$l_6=a_{42}(vc_{51}-a_{55}c_{52})-a_{52}c_{32}m_1.$$

To achieve normal form of the Eq. (15), we make another change of variable i.e. $X=PY$, where

$$Y=(y_1,\quad y_2,\quad y_3,\quad y_4,\quad y_5)^T.$$

Through some algebraic manipulations, Eq. (15) takes the form

$$\dot{Y}=\Pi Y+F, \tag{16}$$

where, $\Pi=P^{-1}AP$ and

$$F=P^{-1}f=\begin{pmatrix}F^1(y_1,y_2,y_3,y_4,y_5)\\F^2(y_1,y_2,y_3,y_4,y_5)\\F^3(y_1,y_2,y_3,y_4,y_5)\\F^4(y_1,y_2,y_3,y_4,y_5)\\F^5(y_1,y_2,y_3,y_4,y_5)\end{pmatrix}.$$

f is given by

$$f=\begin{pmatrix}f^1((y_1,y_2,y_3,y_4,y_5))\\f^2((y_1,y_2,y_3,y_4,y_5))\\f^3((y_1,y_2,y_3,y_4,y_5))\\f^4((y_1,y_2,y_3,y_4,y_5))\\f^5((y_1,y_2,y_3,y_4,y_5))\end{pmatrix},$$

where

$$\begin{aligned}f^1(y_1,y_2,y_3,y_4,y_5)=\;&x_5^*H^3(y_1+y_3+y_4+y_5)^2-\frac{(y_1+y_3+y_4+y_5)^2}{K}-x_5^*H^4(y_1+y_3+y_4+y_5)^3-H^2(y_1+y_3+y_4+y_5)\\&\times(c_{51}y_1+c_{52}y_2+c_{53}y_3+c_{54}y_4+c_{55}y_5)+H^3(y_1+y_3+y_4+y_5)^2(c_{51}y_1+c_{52}y_2+c_{53}y_3+c_{54}y_4\\&+c_{55}y_5),\end{aligned}$$

$$\begin{aligned}f^2(y_1,y_2,y_3,y_4,y_5)=\;&-aw_2(c_{51}y_1+c_{52}y_2+c_{53}y_3+c_{54}y_4+c_{55}y_5)-(c_{21}y_1+c_{22}y_2+c_{23}y_3+c_{24}y_4+c_{25}y_5)(c_{41}y_1\\&+c_{42}y_2+c_{43}y_3+c_{44}y_4+c_{45}y_5)-(x_5^*H^3(y_1+y_3+y_4+y_5)^2-x_5^*H^4(y_1+y_3+y_4+y_5)^3-H^2(y_1\\&+y_3+y_4+y_5)(c_{51}y_1+c_{52}y_2+c_{53}y_3+c_{54}y_4+c_{55}y_5)+H^3(y_1+y_3+y_4+y_5)^2(c_{51}y_1+c_{52}y_2\\&+c_{53}y_3+c_{54}y_4+c_{55}y_5)),\end{aligned}$$

$$f^3(y_1,y_2,y_3,y_4,y_5)=0,$$

$$f^4(y_1,y_2,y_3,y_4,y_5)=b(c_{21}y_1+c_{22}y_2+c_{23}y_3+c_{24}y_4+c_{25}y_5)(c_{41}y_1+c_{42}y_2+c_{43}y_3+c_{44}y_4+c_{45}y_5),$$

$$f^5(y_1,y_2,y_3,y_4,y_5)=e(c_{21}y_1+c_{22}y_2+c_{23}y_3+c_{24}y_4+c_{25}y_5)(c_{51}y_1+c_{52}y_2+c_{53}y_3+c_{54}y_4+c_{55}y_5).$$

Eq. (16) is the normal form of Eq. (15) from which the stability and direction of the Hopf bifurcation can be computed. In Eq. (15), on the right hand side of the first term is linear and the second is non-linear in y's. For evaluating the direction of periodic solution, we can evaluated the following quantities at $r = r_0$ and origin.

$$g_{11} = \frac{1}{4}\left[\frac{\partial^2 F^1}{\partial y_1^2} + \frac{\partial^2 F^1}{\partial y_2^2} + i\left(\frac{\partial^2 F^2}{\partial y_1^2} + \frac{\partial^2 F^2}{\partial y_2^2}\right)\right],$$

$$g_{02} = \frac{1}{4}\left[\frac{\partial^2 F^1}{\partial y_1^2} - \frac{\partial^2 F^1}{\partial y_2^2} - 2\frac{\partial^2 F^2}{\partial y_1 \partial y_2} + i\left(\frac{\partial^2 F^2}{\partial y_1^2} - \frac{\partial^2 F^2}{\partial y_2^2} + 2\frac{\partial^2 F^1}{\partial y_1 \partial y_2}\right)\right],$$

$$g_{20} = \frac{1}{4}\left[\frac{\partial^2 F^1}{\partial y_1^2} - \frac{\partial^2 F^1}{\partial y_2^2} + 2\frac{\partial^2 F^2}{\partial y_1 \partial y_2} + i\left(\frac{\partial^2 F^2}{\partial y_1^2} - \frac{\partial^2 F^2}{\partial y_2^2} - 2\frac{\partial^2 F^1}{\partial y_1 \partial y_2}\right)\right]$$

$$G_{21} = \frac{1}{8}\left[\frac{\partial^3 F^1}{\partial y_1^3} + \frac{\partial^3 F^1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 F^2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 F^2}{\partial y_2^3} + i\left(\frac{\partial^3 F^2}{\partial y_1^3} + \frac{\partial^3 F^2}{\partial y_1 \partial y_2^2} - \frac{\partial^3 F^1}{\partial y_1^2 \partial y_2} - \frac{\partial^3 F^1}{\partial y_2^3}\right)\right],$$

$$G^j_{110} = \frac{1}{2}\left[\frac{\partial^2 F^1}{\partial y_1 \partial y_j} + \frac{\partial^2 F^2}{\partial y_2 \partial y_j} + i\left(\frac{\partial^2 F^2}{\partial y_1 \partial y_j} - \frac{\partial^2 F^1}{\partial y_2 \partial y_j}\right)\right],$$

$$G^j_{101} = \frac{1}{2}\left[\frac{\partial^2 F^1}{\partial y_1 \partial y_j} - \frac{\partial^2 F^2}{\partial y_2 \partial y_j} + i\left(\frac{\partial^2 F^2}{\partial y_1 \partial y_j} + \frac{\partial^2 F^1}{\partial y_2 \partial y_j}\right)\right],$$

$$h^j_{11} = \frac{1}{4}\left[\frac{\partial^2 F^j}{\partial y_1^2} + \frac{\partial^2 F^j}{\partial y_2^2}\right], \quad h^j_{20} = \frac{1}{4}\left[\frac{\partial^2 F^j}{\partial y_1^2} - \frac{\partial^2 F^j}{\partial y_2^2} - 2i\frac{\partial^2 F^j}{\partial y_1 \partial y_2}\right],$$

$$w^j_{11} = \frac{h^j_{11}}{p_j}, \quad w^j_{20} = \frac{h^j_{20}}{(P_j + 2i\upsilon)}, \quad j = 1, 2, 3,$$

and

$$g_{21} = G_{21} + \sum_{j=1}^{3}\left(2G^j_{110}\upsilon^j_{11} + G^j_{101}\upsilon^j_{20}\right).$$

Based on the above analysis, we can see that each g_{ij} can be determined by the parameters. Thus we can compute the following quantities:

$$\begin{aligned}
C_1(0) &= \frac{i}{2\upsilon}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{Re\{C_1(0)\}}{Re\{\lambda'(r_0)\}}, \\
\beta_2 &= 2Re\{C_1(0)\}, \\
T_2 &= -\frac{Im\{C_1(0)\} + \mu_2 Im\{\lambda'(r_0)\}}{\upsilon}.
\end{aligned} \tag{17}$$

Theorem 4.1. *μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $r > r_0$ ($r < r_0$); β_2 determines the stability of bifurcating periodic solutions. The bifurcating periodic solutions are orbitally asymptotically stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions, the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).*

Theorem 4.2. *Let the following inequalities hold:*

$$x_5^*K < (1 + x_1^*), \tag{18}$$
$$3c_3m_2x_5^* < c_2m_1x_3^*, \tag{19}$$
$$\left(\frac{K\theta}{\theta_1}(c_3e + 1 + a)\right)^2 < \frac{c_3m_2x_4^*x_1^*}{6x_2^*(1 + x_1^*)}, \tag{20}$$
$$\left(\frac{K\theta}{\theta_1}(c_2b - 1)\right)^2 < \frac{4c_2x_5^*x_1^*m_1x_3^*c_2}{9x_4^*x_2^*(1 + x_1^*)}, \tag{21}$$

with

$$\frac{3Kx_5^*x_2^*}{2x_1^*((1 + x_1^*) - Kx_5^*)} < c_1 < \frac{c_3m_2x_4^*}{2x_5^*}\left(\frac{1}{K} - \frac{x_5^*}{(1 + x_1^*)}\right), \tag{22}$$

$$0 < c_2 < \frac{2d_1x_3^*}{3m_1x_4^*}, \tag{23}$$

$$c_3 > \frac{rx_3^*}{m_2x_4^*}. \tag{24}$$

Then E^* is globally asymptotically stable with respect to solutions initiating in the interior of the region Ω.

Proof. We consider the following positive definite function about E^*

$$V(x_1, x_2, x_3, x_4, x_5) = c_1(x_1 - x_1^* - x_1^* ln\frac{x_1}{x_1^*}) + \frac{1}{2}(x_2 - x_2^*)^2 + \frac{1}{2}(x_3 - x_3^*)^2 + \frac{c_2}{2}(x_4 - x_4^*)^2 + \frac{c_3}{2}(x_5 - x_5^*)^2. \tag{25}$$

Then the derivative along solutions, $\dot{V}$ is given by

$$\begin{aligned}\dot{V} = {} & c_1(x_1 - x_1^*)\left(1 - \frac{x_1}{K} - \frac{x_5}{(1 + x_1)}\right) + (x_3 - x_3^*)(rx_5 - d_1x_3) + (x_2 - x_2^*)\left(\frac{x_1x_5}{(1 + x_1)} - ax_2x_5 - x_2x_4 - cx_2\right) \\ & + c_2(m_1x_3 + bx_2x_4 - d_2x_4)(x_4 - x_4^*) + c_3(x_5 - x_5^*)(m_2x_4 + ex_2x_5 - d_3x_5).\end{aligned} \tag{26}$$

After some algebraic manipulation, this can be written as

$$\begin{aligned}\dot{V} = {} & -c_1z_1^2\left(\frac{1}{K} - \frac{x_5^*}{(1 + x_1)(1 + x_1^*)}\right) + \frac{x_5^*}{(1 + x_1)(1 + x_1^*)}z_1z_2 - \frac{c_1}{(1 + x_1)}z_1z_5 - \frac{x_5^*x_1^*}{x_2^*(1 + x_1^*)}z_2^2 - d_1z_3^2 - \frac{m_1x_3^*c_2}{x_4^*}z_4^2 \\ & - \frac{m_2x_4^*c_3}{x_5^*}z_5^2 + \left(c_3ex_5 + \frac{x_1}{(1 + x_1)} - ax_2\right)z_2z_5 + (c_2bx_4 - x_2)z_2z_4 + c_2m_1z_3z_4 + r_1z_3z_5 + c_3m_2z_4z_5,\end{aligned} \tag{27}$$

where $z_1 = (x_1 - x_1^*),\ z_2 = (x_2 - x_2^*),\ z_3 = (x_3 - x_3^*),\ z_4 = (x_4 - x_4^*),\ z_5 = (x_5 - x_5^*)$. Hence $\dot{V}$ can be written as the sum of the quadratics

$$\begin{aligned}\dot{V} = {} & -\frac{1}{2}s_{11}z_1^2 + s_{12}z_1z_2 - \frac{1}{2}s_{22}z_2^2 - \frac{1}{2}s_{11}z_1^2 + s_{15}z_1z_5 - \frac{1}{2}s_{22}z_2^2 - \frac{1}{2}s_{22}z_2^2 + s_{25}z_2z_5 - \frac{1}{2}s_{55}z_5^2 - \frac{1}{2}s_{22}z_2^2 + s_{24}z_2z_4 \\ & - \frac{1}{2}s_{44}z_4^2 - \frac{1}{2}s_{33}z_3^2 + s_{34}z_2z_4 - \frac{1}{2}s_{44}z_4^2 - \frac{1}{2}s_{33}z_3^2 + s_{35}z_2z_4 - \frac{1}{2}s_{55}z_4^2 - \frac{1}{2}s_{33}z_3^2 + s_{35}z_2z_4 - \frac{1}{2}s_{55}z_4^2,\end{aligned} \tag{28}$$

where

$$s_{11} = c_1\left(\frac{1}{K} - \frac{x_5^*}{(1 + x_1)(1 + x_1^*)}\right), \quad s_{22} = \frac{2x_5^*x_1^*}{3x_2^*(1 + x_1^*)}, \quad s_{33} = d_1,$$
$$s_{44} = \frac{2m_1x_3^*c_2}{3x_4^*}, \quad s_{55} = \frac{m_2x_4^*c_3}{2x_5^*}, \quad s_{15} = -\frac{c_1}{(1 + x_1)},$$
$$s_{25} = \left(c_3ex_5 + \frac{x_1}{(1 + x_1)} - ax_2\right), \quad s_{24} = (c_2bx_4 - x_2), \quad s_{34} = c_2m_1,$$
$$s_{12} = \frac{x_5^*}{(1 + x_1)(1 + x_1^*)}, \quad s_{35} = r_1s_{45} = c_3m_2.$$

Sufficient conditions for $\dot{V}$ to be negative definite are that the following conditions hold:

$$s_{12}^2 < s_{11}s_{22}, s_{11} > 0, \tag{29}$$
$$s_{15}^2 < s_{11}s_{55}, s_{11} > 0, \tag{30}$$
$$s_{25}^2 < s_{55}s_{22}, \tag{31}$$
$$s_{24}^2 < s_{44}s_{22}, \tag{32}$$
$$s_{34}^2 < s_{33}s_{44}, \tag{33}$$
$$s_{35}^2 < s_{33}s_{55}, \tag{34}$$
$$s_{45}^2 < s_{44}s_{55}, . \tag{35}$$

We note that inequalities (29), (30), (33) and (34) are automatically satisfied for the suitable values of $c_1,\ c_2,\ c_3$ given by (18), (22)–(24). However (20) implies (31), (21) implies (32) and (19) implies (35). Hence $\dot{V}$ is negative definite under the condition Eq. (15), and so V is a Liapunov function with respect to E^* whose domains contains Ω, proving the theorem. □

5. Numerical results

We substantiate all the previous analytical findings with the help of numerical simulations performed with Matlab. The stability of the first equilibria $E_A \equiv (0.7, 0, 0, 0, 0)$ can be seen in Fig. 1. It is obtained for the parameter values

$$k = 0.7,\quad a = 0.4,\quad c = 0.01,\quad r = 0.01,\quad d_1 = 0.02,\quad m_1 = 0.015,$$
$$b = 0.8,\quad d_2 = 0.9,\quad m_2 = 0.8,\quad d_3 = 0.54,\quad e = 0.3.$$

In this case the system has axial equilibrium point in which system is asymptotically stable.

The interior equilibrium is shown to be stable for the following parameter values

$$k = 10,\quad a = 0.4,\quad c = 0.001,\quad r = 0.2,\quad d_1 = 0.11,\quad m_1 = 0.1,$$
$$b = 0.8,\quad d_2 = 0.55,\quad m_2 = 0.35,\quad e = 0.3,\quad d_3 = 0.6,$$

In Fig. 2. For this choice of parameter values the unique interior equilibrium point $E^* \equiv (5.8975, 0.5096, 5.1449, 3.6149, 2.8297)$ is asymptotically stable. We studied the Hopf-bifurcation of the system taking r as the bifurcation parameter, the transversality condition hold with these parameters when $r = r_0 = 0.244$. From Fig. 3, it is clear that system is stable in $(0, r_0)$ and when $r \geqslant r_0$, then the system becomes unstable and Hopf bifurcation occurs. Further, we have also determined the stability and direction of Hopf-bifurcating periodic solutions at the critical value r_0. For the same parameteric values and $r = r_0 = 0.244$, we have calculated the following values

$$g_{11} = -0.0341521 - 0.0149509i,\quad g_{20} = -0.0309595 - 0.0182149i,$$
$$g_{02} = -0.0374306 - 0.0117438i,\quad g_{21} = 0.0018869 + 0.00342361i,$$
$$C_1(0) = -0.00284741 - 0.00705047i,\quad u'(r_0) = 0.0031527.$$

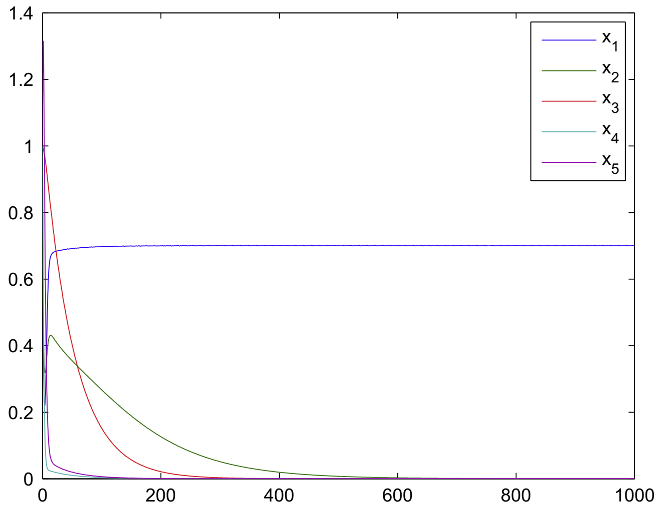

Fig. 1. The axial equilibrium point $E_A(0.7,\ 0,\ 0,\ 0,\ 0)$ of the system is asymptotically stable.

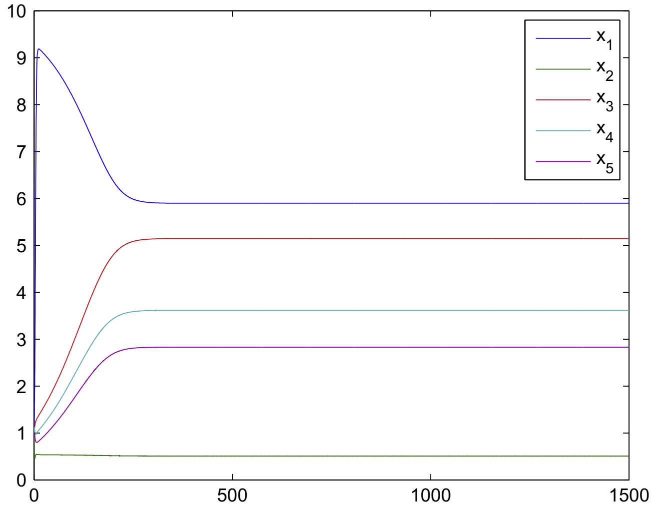

Fig. 2. The interior equilibrium point $E^*(5.8975, 0.5096, 5.1449, 3.6149, 2.8297)$ of the system is asymptotically stable.

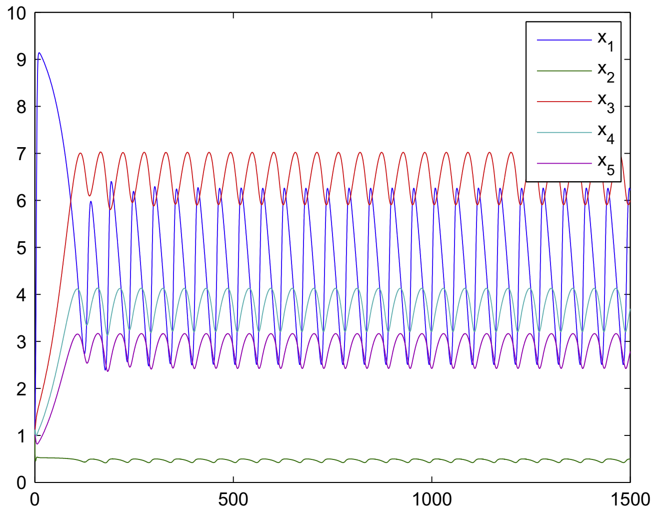

Fig. 3. When $r = .25 \geqslant r_0 = .244$ and remaining parameters have same value, then the positive interior equilibrium point E^* of system looses its stability and a Hopf-bifurcation occurs.

It follows from (17) that $\mu_2 > 0$ and $\beta_2 < 0$. Therefore, the bifurcation periodic solutions exist for $r > r_0$ and the corresponding periodic solutions are orbitally asymptotically stable.

6. Conclusion

In this paper, we have proposed a mathematical model to study the stability and Hopf-bifurcation analysis of a prey–predator system incorporating age based predation. We have studied the stability behaviour of the system around the feasible steady states. Our theoretical as well as numerical results show that for a certain threshold of the system parameters, the system possesses asymptotic stability around the positive interior equilibrium depicting the co-existence of all the species. Further, from the stability analysis and numerical simulation, it is also concluded that the prey population will survive and predator population will go to extinction. From qualitative and numerical analysis we find that r is a bifurcating parameter for which the interior equilibrium point shows stable oscillatory behaviour when $r \geqslant r_0$.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.apm.2013.01.036.

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