EXISTENCE OF A POSITIVE SOLUTION TO A RIGHT FOCAL BOUNDARY VALUE PROBLEM

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Abstract. In this paper we apply the recent extension of the Leggett-Williams Fixed Point Theorem which requires neither of the functional boundaries to be invariant to the second order right focal boundary value problem. We demonstrate a technique that can be used to deal with a singularity and provide a non-trivial example.

1. Introduction

The recent topological proof and extension of the Leggett-Williams fixed point theorem [3] does not require either of the functional boundaries to be invariant with respect to a functional wedge and its proof uses topological methods instead of axiomatic index theory. Functional fixed point theorems (including [2, 4, 5, 6, 8]) can be traced back to Leggett and Williams [7] when they presented criteria which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant with regard to the concave functional boundary of a functional wedge. Avery, Henderson, and O’Regan [1], in a dual of the Leggett-Williams fixed point theorem, gave conditions which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant relative to the concave functional boundary of a functional wedge. We will demonstrate a technique to take advantage of the added flexibility of the new fixed point theorem for a right focal boundary value problem.

2. Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

Definition 1. Let $E$ be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by

$x \leq y$ if and only if $y - x \in P$.

Definition 2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

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**Definition 3.** A map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map $\beta$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let $\alpha$ and $\psi$ be non-negative continuous concave functionals on $P$ and $\delta$ and $\beta$ be non-negative continuous convex functionals on $P$; then, for non-negative real numbers $a$, $b$, $c$ and $d$, we define the following sets:

1. $$A := A(\alpha, \beta, a, d) = \{ x \in P : a \leq \alpha(x) \text{ and } \beta(x) \leq d \},$$
2. $$B := B(\alpha, \delta, \beta, a, b, d) = \{ x \in A : \delta(x) \leq b \},$$
3. $$C := C(\alpha, \psi, \beta, a, c, d) = \{ x \in A : c \leq \psi(x) \}.$$

We say that $A$ is a functional wedge with concave functional boundary defined by the concave functional $\alpha$ and convex functional boundary defined by the convex functional $\beta$. We say that an operator $T : A \to P$ is invariant with respect to the concave functional boundary, if $a \leq \alpha(Tx)$ for all $x \in A$, and that $T$ is invariant with respect to the convex functional boundary, if $\beta(Tx) \leq d$ for all $x \in A$. Note that $A$ is a convex set. The following theorem is an extension of the original Leggett-Williams fixed point theorem [7].

**Theorem 4. [Extension of Leggett-Williams]** Suppose $P$ is a cone in a real Banach space $E$, $\alpha$ and $\psi$ are non-negative continuous concave functionals on $P$, $\delta$ and $\beta$ are non-negative continuous convex functionals on $P$, and for non-negative real numbers $a$, $b$, $c$ and $d$ the sets $A$, $B$ and $C$ are as defined in (1), (2) and (3). Furthermore, suppose that $A$ is a bounded subset of $P$, that $T : A \to P$ is completely continuous and that the following conditions hold:

(A1) $\{ x \in A : c < \psi(x) \text{ and } \delta(x) < b \} \neq \emptyset$ and $\{ x \in P : \alpha(x) < a \text{ and } d < \beta(x) \} = \emptyset$;
(A2) $\alpha(Tx) \geq a$ for all $x \in B$;
(A3) $\alpha(Tx) \geq a$ for all $x \in A$ with $\delta(Tx) > b$;
(A4) $\beta(Tx) \leq d$ for all $x \in C$; and,
(A5) $\beta(Tx) \leq d$ for all $x \in A$ with $\psi(Tx) < c$.

Then $T$ has a fixed point $x^* \in A$.

3. **Right Focal Boundary Value Problem**

In this section we will illustrate the key techniques for verifying the existence of a positive solution for a boundary value problem using the newly developed extension of the Leggett-Williams fixed point theorem, applying the properties of a Green's function, bounding the
nonlinearity by constants over some intervals, and using concavity to deal with a singularity. Consider the second order nonlinear focal boundary value problem

\begin{equation}
 x''(t) + f(x(t)) = 0, \quad t \in (0, 1),
\end{equation}

\begin{equation}
 x(0) = 0 = x'(1),
\end{equation}

where \( f : \mathbb{R} \to [0, \infty) \) is continuous. If \( x \) is a fixed point of the operator \( T \) defined by

\[
 Tx(t) := \int_0^1 G(t,s)f(x(s))ds,
\]

where

\[
 G(t,s) = \begin{cases} 
 t & : t \leq s, \\
 s & : s \leq t,
\end{cases}
\]

is the Green's function for the operator \( L \) defined by

\[
 Lx(t) := -x''(t),
\]

with right-focal boundary conditions

\[
 x(0) = 0 = x'(1),
\]

then it is well known that \( x \) is a solution of the boundary value problem (4), (5). Throughout this section of the paper we will use the facts that \( G(t,s) \) is nonnegative, and for each fixed \( s \in [0, 1] \), the Green's function is nondecreasing in \( t \).

Define the cone \( P \subset E = C[0,1] \) by

\[
 P := \{ x \in E : x \text{ is nonnegative, nondecreasing, and concave} \}.
\]

For fixed \( \nu, \tau, \mu \in [0,1] \) and \( x \in P \), define the concave functionals \( \alpha \) and \( \psi \) on \( P \) by

\[
 \alpha(x) := \min_{t \in [\tau,1]} x(t), \quad \psi(x) := \min_{t \in [\mu,1]} x(t) = x(\mu),
\]

and the convex functionals \( \delta \) and \( \beta \) on \( P \) by

\[
 \delta(x) := \max_{t \in [0,\nu]} x(t), \quad \beta(x) := \max_{t \in [0,1]} x(t) = x(1).
\]

In the following theorem, we demonstrate how to apply the Extension of the Leggett-Williams Fixed Point Theorem (Theorem 4), to prove the existence of at least one positive solution to (4), (5).

**Theorem 5.** If \( \tau, \nu, \mu \in (0,1) \) are fixed with \( \tau \leq \mu < \nu \leq 1 \), \( d \) and \( m \) are positive real numbers with \( 0 < m \leq d\mu \) and \( f : [0, \infty) \to [0, \infty) \) is a continuous function such that

\[
 (a) \quad f(w) \geq \frac{d}{\nu-d} w \quad \text{for} \quad w \in [\tau d, \nu d],
\]

\[
 (b) \quad f(w) \text{ is decreasing for} \quad w \in [0, m] \quad \text{with} \quad f(m) \geq f(w) \quad \text{for} \quad w \in [m, d], \quad \text{and}
\]

\[
 (c) \quad \int_0^\mu s f \left( \frac{ms}{\mu} \right) ds \leq \frac{2d-f(m)(1-\mu^2)}{2},
\]

then the operator \( T \) has at least one positive solution \( x^* \in A(\alpha, \beta, \tau d, \mu) \).
Proof. Let $a = \tau d$, $b = \nu d = \frac{a}{\nu}$, and $c = d\mu$. Let $x \in A(\alpha, \beta, a, d)$ then if $t \in (0, 1)$, by the properties of the Green’s function $(Tx)'(t) = -f(x(t))$ and $Tx(0) = 0 = (Tx)'(1)$, thus

$$T : A(\alpha, \beta, a, d) \rightarrow P.$$ 

We will also take advantage of the following property of the Green’s function. For any $y, w \in [0, 1]$ with $y \leq w$ we have

$$\min_{s \in [0, 1]} \frac{G(y, s)}{G(w, s)} \geq \frac{y}{w}. \tag{6}$$ 

By the Arzela-Ascoli Theorem it is a standard exercise to show that $T$ is a completely continuous operator using the properties of $G$ and $f$, and by the definition of $\beta$, we have that $A$ is a bounded subset of the cone $P$. Also, if $x \in P$ and $\beta(x) > d$, then by the properties of the cone $P$,

$$\alpha(x) = x(\tau) \geq \left(\frac{\tau}{1}\right) x(1) = \tau \beta(x) > \tau d = a.$$ 

Therefore,

$$\{x \in P : \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset.$$ 

For any $K \in \left(\frac{2d}{2-\mu}, \frac{2d}{2-\nu}\right)$ the function $x_K$ defined by

$$x_K(t) \equiv \int_0^1 KG(t, s)ds = \frac{K(2-t)}{2} \in A,$$ 

since

$$\alpha(x_K) = x_K(\tau) = \frac{K\tau(2-\tau)}{2} > \frac{d\tau(2-\tau)}{2-\mu} \geq d\tau = a,$$

$$\beta(x_K) = x_K(1) = \frac{K}{2} < \frac{d}{2-\nu} \leq d,$$

and $x_K$ has the properties that

$$\psi(x_K) = x_K(\mu) = \frac{K\mu(2-\mu)}{2} > \left(\frac{2d}{2-\mu}\right) \left(\frac{\mu(2-\mu)}{2-\mu}\right) = d\mu = c$$

and

$$\delta(x_K) = x_K(\nu) = \frac{K\nu(2-\nu)}{2} < \left(\frac{2d}{2-\nu}\right) \left(\frac{\nu(2-\nu)}{2-\nu}\right) = d\nu = b.$$ 

Hence

$$\{x \in A : c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset.$$ 

Claim 1: $\alpha(Tx) \geq a$ for all $x \in B$.

Let $x \in B$. Thus by condition (a),

$$\alpha(Tx) = \int_0^1 G(\tau, s) f(x(s)) ds \geq \left(\frac{a}{\tau(\nu-\tau)}\right) \int_\tau^\nu G(\tau, s) ds$$

$$= \left(\frac{a}{\tau(\nu-\tau)}\right) (\tau(\nu-\tau)) = a.$$ 

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Claim 2: $\alpha(Tx) \geq a$, for all $x \in A$ with $\delta(Tx) > b$.

Let $x \in A$ with $\delta(Tx) > b$. Thus by the properties of $G(6)$,

$$
\alpha(Tx) = \int_0^1 G(\tau, s) f(x(s)) \, ds \geq \left( \frac{\tau}{\nu} \right) \int_0^1 G(\nu, s) f(x(s)) \, ds
$$

$$
= \left( \frac{\tau}{\nu} \right) \delta(Tx) \left( \frac{\tau}{\nu} \right) (d\nu) = a.
$$

Claim 3: $\beta(Tx) \leq d$, for all $x \in C$.

Let $x \in C$, thus by the concavity of $x$, for $s \in [0, \mu]$ we have

$$
x(s) \geq \frac{cs}{\mu} \geq \frac{ms}{\mu}.
$$

Hence by properties (b) and (c),

$$
\beta(Tx) = \int_0^1 G(1, s) f(x(s)) \, ds = \int_0^1 s f(x(s)) \, ds
$$

$$
= \int_0^\mu s f(x(s)) \, ds + \int_{\mu}^1 s f(x(s)) \, ds
$$

$$
\leq \int_0^\mu s f\left( \frac{ms}{\mu} \right) \, ds + f(m) \int_{\mu}^1 s \, ds
$$

$$
\leq \frac{2d - f(m)(1 - \mu^2)}{2} + \frac{f(m)(1 - \mu^2)}{2} = d.
$$

Claim 4: $\beta(Tx) \leq d$, for all $x \in A$ with $\psi(Tx) < c$.

Let $x \in A$ with $\psi(Tx) < c$. Thus by the properties of $G(6)$,

$$
\beta(Tx) = \int_0^1 G(1, s) f(x(s)) \, ds \leq \left( \frac{1}{\mu} \right) \int_0^1 G(\mu, s) f(x(s)) \, ds
$$

$$
= \left( \frac{1}{\mu} \right) Tx(\mu) = \left( \frac{1}{\mu} \right) \psi(Tx) \leq \left( \frac{1}{\mu} \right) c = d.
$$

Therefore, the hypotheses of Theorem 4 have been satisfied; thus the operator $T$ has at least one positive solution $x^* \in A(\alpha, \beta, a, d)$. \hfill \Box

We note that because of the concavity of solutions, the proof of Theorem 5 remains valid for certain singular nonlinearities as presented in this example.

**Example:** Let

$$
d = \frac{5}{4}, \tau = \frac{1}{16}, \mu = \frac{3}{4}, \text{ and } \nu = \frac{15}{16}.
$$

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Then the boundary value problem

\[ x'' + \frac{1}{\sqrt{x}} + \sqrt{x} = 0, \]

with right-focal boundary conditions

\[ x(0) = 0 = x'(1), \]

has at least one positive solution \( x^* \) which can be verified by the above theorem, with

\[ \frac{5}{64} \leq x^*(1/16) \quad \text{and} \quad x^*(1) \leq \frac{5}{4}. \]

**References**


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