Interval oscillation criteria for second-order forced delay dynamic equations with mixed nonlinearities

Ravi P. Agarwal a,b,∗, Douglas R. Anderson c, Ağacık Zafer d

a Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901-6975, USA
b Mathematics and Statistics Department, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
c Department of Mathematics and Computer Science, Concordia College, Moorhead, MN 56562, USA
d Department of Mathematics, Middle East Technical University, 06531, Ankara, Turkey

Interval oscillation criteria are established for second-order forced delay dynamic equations on time scales containing mixed nonlinearities of the form

\[(r(t)\Phi_\alpha(x^N(t)))^\Delta + p_0(t)\Phi_\omega(x(\tau_0(t))) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(\tau_i(t))) = e(t), \quad t \in [t_0, \infty)_\mathbb{T}\]

where \(\mathbb{T}\) is a time scale, \(t_0 \in \mathbb{T}\) a fixed number, \([t_0, \infty)_\mathbb{T}\) is a time scale interval; \(\Phi_\omega(u) = |u|^{\alpha-1}u\); the functions \(r, p_i, e : [t_0, \infty)_\mathbb{T} \to \mathbb{R}\) are right-dense continuous with \(r > 0\) nondecreasing; \(\tau_k : \mathbb{T} \to \mathbb{T}\) are nondecreasing right-dense continuous with \(\tau_k(t) \leq t\); \(\lim_{t \to \infty} \tau_k(t) = \infty\); and the exponents satisfy

\[\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots \beta_n > 0.\]

All results are new even for \(\mathbb{T} = \mathbb{R}\) and \(\mathbb{T} = \mathbb{Z}\).

Analogous results for related advance type equations are also given, as well as extended delay and advance equations. The theory can be applied to second-order dynamic equations regardless of the choice of delta or nabla derivatives. Two examples are provided to illustrate one of the theorems.

© 2009 Elsevier Ltd. All rights reserved.

1. Introduction

Following Hilger’s landmark paper [1], a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus, where a time scale is simply any nonempty closed set of real numbers. The oscillation theory has been developed very rapidly since the discovery of time scale calculus with this understanding. For some papers on the subject, we refer to [2–16] and the references cited therein. Throughout the paper it is assumed that the reader is familiar with time scale calculus. For an introduction to time scale calculus and dynamic equations, we refer to the seminal books by Bohner and Peterson [9,17].

∗ The financial support by TUBITAK to complete this work is greatly appreciated.

∗ Corresponding author at: Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, FL 32901-6975, USA.
E-mail addresses: agarwal@fit.edu (R.P. Agarwal), andersod@cord.edu (D.R. Anderson), zafer@metu.edu.tr (A. Zafer).

0898-1221/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2009.09.010
In this paper we consider the second-order nonlinear delay dynamic equation with forcing term
\[
(r(t)\Phi_a(x^d(t)))^n + p_0(t)\Phi_a(x(\tau(t))) + \sum_{i=1}^n p_i(t)\Phi_{\beta_i}(x(\tau_i(t))) = e(t), \quad t \in [t_0, \infty)_\tau
\]  
(1.1)
and without forcing term
\[
(r(t)\Phi_a(x^d(t)))^n + p_0(t)\Phi_a(x(\tau(t))) + \sum_{i=1}^n p_i(t)\Phi_{\beta_i}(x(\tau_i(t))) = 0, \quad t \in [t_0, \infty)_\tau,
\]  
(1.2)
where \(\tau\) is a time scale (a closed nonempty subset of real numbers) which is unbounded from above, with \(t_0 \in \mathbb{T}\) a fixed number; \([t_0, \infty)_\tau\) denotes the time scale interval \([t_0, \infty) \cap \mathbb{T}\); \(\Phi_a(u) = |u|^{\alpha-1}u\); the functions \(r, p, e : \mathbb{T} \to \mathbb{R}\) are right-dense continuous with \(r > 0\) nondecreasing; the delays \(\tau_k : \mathbb{T} \to \mathbb{T}\) are nondecreasing and right-dense continuous with \(\tau_k(t) \leq t\) and \(\lim_{t \to \infty} \tau_k(t) = \infty\); and the exponents satisfy
\[\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots \beta_n > 0.\]

We restrict our consideration to those solutions of Eq. (1.1) which exist on the time scale half-line \([t_k, \infty)_\tau\), where \(t_k\) may depend on the particular solution, a nontrivial function in any neighborhood of infinity. As usual, a solution \(x(t)\) of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. The equation is called oscillatory if every proper solution is oscillatory [18–20].

To the best of our knowledge, the first study concerning the oscillation of equations with mixed nonlinearities is performed by Sun and Wong [21] for second-order forced differential equations of the form
\[
x^n + p_0(t)x + \sum_{i=1}^n p_i(t)\Phi_{\beta_i}(x) = e(t), \quad t \geq t_0,
\]  
(1.3)
where \(\beta_1 > \cdots > \beta_m > 1 > \beta_{m+1} > \cdots \beta_n > 0\). The authors obtained interval oscillation criteria for Eq. (1.3) by using an arithmetic–geometric inequality and employing arguments developed earlier in [22–24]. Very recently, Sun and Meng [25] have studied the same equation by making use of some of the arguments developed by Kong [26]. As stated in [21], equations with mixed nonlinearities arise in the growth of bacteria population with competitive species and therefore require more attention.

Note that Eq. (1.3) is a special case of
\[
(\Phi_a(x'))' + p_0(t)\alpha(x') + \sum_{i=1}^n p_i(t)\Phi_{\beta_i}(x) = e(t), \quad t \geq t_0,
\]  
(1.4)
where \(\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots \beta_n > 0\).

In [21], Agarwal and Zafer extended the results in [21] to dynamic equations on time scales of the form
\[
(r(t)\Phi_a(x^d)) + p_0(t)\Phi_a(x) + \sum_{i=1}^n p_i(t)\Phi_{\beta_i}(x) = e(t), \quad t \in [t_0, \infty)_\tau,
\]  
(1.5)
where \(\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots \beta_n > 0\), and obtained interval oscillation criteria for Eq. (1.4) and in particular for Eq. (1.3) as in the special case \(\mathbb{T} = \mathbb{R}\).

In a recent paper, Anderson and Zafer [7] established interval criteria for oscillation of second-order forced nonlinear dynamic equations with delay and advance terms given by
\[
(r(t)\Phi_a(x^d(t))) + p(t)\Phi_{\rho}(x(\tau(t))) + q(t)\Phi_{\gamma}(x(\theta(t))) = e(t), \quad t \in [t_0, \infty)_\tau,
\]  
(1.6)
which include as special cases the delay equation
\[
(r(t)\Phi_a(x^d(t))) + p(t)\Phi_{\rho}(x(t)) = e(t)
\]  
(1.7)
and the advance equation
\[
(r(t)\Phi_a(x^d(t))) + q(t)\Phi_{\gamma}(x(\theta(t))) = e(t).
\]  
(1.8)

For some related work in the continuous case, we may refer to Sun [27] for Eq. (1.7) when \(\alpha = 1\) and to Zafer [28] for Eq. (1.6). The results given by Sun in [27] extend those of El-Sayed [22], Nasr [23] and Wong [24] to the delay differential equations case. More interval oscillation criteria can be found in [29–34].

In the present work we continue our investigation to extend the work in [4] to delay dynamic equations with mixed nonlinearities and several delays of the form (1.1).

The paper is organized as follows. The next Section contains some lemmas which we rely on in later sections. The main results are given in Section 3. In Section 4 we show that the theory can be applied to second-order delay dynamic equations.
regardless of the choice of delta or nabla derivatives. Section 5 illustrates the usefulness of the results obtained in Section 3, where we restate the main results for the special cases $\mathbb{T} = \mathbb{R}$ (differential equations), $\mathbb{T} = \mathbb{Z}$ (difference equations), and $\mathbb{T} = q^\mathbb{N}$ ($q$-difference equations). In Section 6 we show that the method can be applied to advance type equations with mixed nonlinearities. We show in Section 7 how the extended delay and advance functions can be used to remove the restrictions that the range of the delay and advance functions must belong to $\mathbb{T}$. Finally, we provide two examples in Section 8 illustrating one of theorems.

2. Foundational lemmas

We need the following preparatory lemmas. The first two are simple extensions of [21, Lemma 1]; see also [4]. Lemma 2.4 is essential for our work.

**Lemma 2.1.** For any given $n$-tuple $\{\beta_1, \beta_2, \ldots, \beta_n\}$ satisfying

$\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0,$

there corresponds an $n$-tuple $\{\eta_1, \eta_2, \ldots, \eta_n\}$ such that

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \quad \sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1.$$  \hfill (2.1)

If $n = 2$ and $m = 1$, we may take (cf. [21] for $\alpha = 1$)

$$\eta_1 = \frac{\alpha - \beta_2(1 - \eta_0)}{\beta_1 - \beta_2}, \quad \eta_2 = \frac{\beta_1(1 - \eta_0) - \alpha}{\beta_1 - \beta_2},$$

where $\eta_0$ is any positive number with $\beta_1 \eta_0 < \beta_1 - \alpha$.

**Lemma 2.2.** For any given $n$-tuple $\{\beta_1, \beta_2, \ldots, \beta_n\}$ satisfying

$\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0,$

there corresponds an $n$-tuple $\{\eta_1, \eta_2, \ldots, \eta_n\}$ such that

$$\sum_{i=1}^{n} \beta_i \eta_i = \alpha, \quad \sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1.$$  \hfill (2.2)

If $n = 2$ and $m = 1$, it turns out that

$$\eta_1 = \frac{\alpha - \beta_2}{\beta_1 - \beta_2}, \quad \eta_2 = \frac{\beta_1 - \alpha}{\beta_1 - \beta_2}.$$

**Lemma 2.3** (Young’s Inequality). If $p > 1$ and $q > 1$ are conjugate numbers ($\frac{1}{p} + \frac{1}{q} = 1$), then for any $u, v \in \mathbb{R}$,

$$\frac{|u|^p}{p} + \frac{|v|^q}{q} \geq |uv|,$$

where equality holds if and only if $u = |v|^{q-2}v$.

Let $\gamma$ and $\delta$ be positive real numbers with $\gamma > \delta$. Put $u = A^{\delta/\gamma}\chi^\delta, p = \gamma/\delta$, and $v = (B\alpha)^{1-\delta/\gamma}(\gamma - \delta)^{\delta/\gamma - 1}$. It follows from Lemma 2.3 that

$$Ax^{\gamma} + B \geq \gamma \delta^{-\delta/\gamma}(\gamma - \delta)^{(\delta/\gamma) - 1}A^{\delta/\gamma}B^{1-\delta/\gamma}\chi^\delta$$

(2.3)

for all $A, B, x \geq 0$. Rewriting the above inequality we also have

$$Cx^\delta - D \leq \delta^{-\delta/\gamma}(\gamma - \delta)^{(\delta/\gamma) - 1}C^{1-\delta/\gamma}D^{1-\delta/\gamma}\chi^\gamma$$

(2.4)

for all $C, x \geq 0$ and $D > 0$.

**Lemma 2.4.** Let $\tau : \mathbb{T} \to \mathbb{T}$ be a nondecreasing right-dense continuous function with $\tau(t) \leq t$, and $a, b \in \mathbb{T}$ with $a < b$. If $x \in C^\gamma_\alpha[a, b]_\mathbb{T}$ is a positive function for which $r(t)\Phi_\alpha(x^\gamma(t))$ is nonincreasing on $[\tau(a), b]_\mathbb{T}$, then

$$\frac{x(\tau(t))}{x^\gamma(t)} \geq \frac{\tau(t) - \tau(a)}{\sigma(t) - \tau(a)}, \quad t \in [a, b]_\mathbb{T}.$$  \hfill (2.5)
The proof is based on arguments developed in [5]. By the Mean Value Theorem [17, Theorem 1.14],
\[ x(t) - x(t(a)) \geq x^3(\xi)(t - t(a)) \]
for some \( \xi \in (t(a), t]_\tau \), for any \( t \in (t(a), b]_\tau \). We may claim that \( x^3(t) \leq x^3(\xi) \). If the claim is true, then
\[ x(t) - x(t(a)) \geq x^3(t)(t - t(a)), \quad t \in [t(a), b]_\tau. \tag{2.6} \]
Indeed, since \( r(t)\Phi_a(x^3(t)) \) is nonincreasing and \( r(t) \) is nondecreasing by assumption, we have
\[ r(t)\Phi_a(x^3(t)) \leq r(\xi)\Phi_a(x^3(\xi)) \leq r(t)\Phi_a(x^3(\xi)), \quad t > \xi. \]
Since \( \Phi_a^{-1} \) is increasing, the claim follows.

Next we define
\[ \eta(s) := x(s) - (s - t(a))x^3(s), \quad s \in [t(t), \sigma(t)]_\tau, \quad t \in [a, b]_\tau. \]

From (2.6) we have \( \eta(s) \geq x(t(a)) > 0 \) for \( s \in [t(t), \sigma(t)]_\tau \) and \( t \in (a, b]_\tau \). Thus, we have
\[ 0 < \int_{t(t)}^{\sigma(t)} \frac{\eta(s)}{x(s)x^\sigma(s)} \Delta s = \int_{t(t)}^{\sigma(t)} \left( \frac{s - t(a)}{x(s)} \right)^\Delta \Delta s = \frac{\sigma(t) - t(a)}{x^\sigma(t)} - \frac{\sigma(t) - t(a)}{x(t)}, \]
and hence inequality (2.5). \( \square \)

3. Main results

Following [5,7,28], we define for \( a, b \in [t_0, \infty)_\tau \) with \( a < b \) the admissible set
\[ \mathcal{A}(a, b) := \left\{ u \in C^1_{ad}[a, b]_\tau : u(a) = 0 = u(b), u \not\equiv 0 \right\}. \]

Theorem 3.1. Suppose that for any given \( T \in [t_0, \infty)_\tau \) there exist subintervals \([a_1, b_1]_\tau\) and \([a_2, b_2]_\tau\) of \([T, \infty)_\tau\) such that
\[ p_i(t) \geq 0 \quad \text{for} \quad t \in [\hat{a}_1, b_1]_\tau \cup [\hat{a}_2, b_2]_\tau, \quad (i = 0, 1, 2, \ldots, n) \tag{3.1} \]
and
\[ (-1)^ke(t) \geq 0 \quad (\not\equiv 0) \quad \text{for} \quad t \in [\hat{a}_k, b_k]_\tau, \quad (k = 1, 2), \tag{3.2} \]
where
\[ \hat{a}_k = \min\{\tau_i(a_k) : i = 0, 1, 2, \ldots, n\}. \tag{3.3} \]
Let \( \{\eta_i\}, i = 1, 2, \ldots, n, \) be an \( n \)-tuple satisfying (2.1) in Lemma 2.1. If there exists a function \( u \in \mathcal{A}(a_k, b_k), (k = 1, 2), \) such that
\[ \int_{a_k}^{b_k} \left\{ |u^\sigma(t)|^{\alpha+1}|p(t) - |u^\Delta(t)|^{\alpha+1}r(t)\right\} \Delta t \geq 0 \tag{3.4} \]
for \( k = 1, 2, \) where
\[ p(t) = p_0(t) \left[ \frac{\tau_0(t) - \tau_0(a_k)}{\sigma(t) - \tau_0(a_k)} \right]^\alpha + \eta|e(t)|^\eta_0 \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{\tau_i(t) - \tau_i(a_k)}{\sigma(t) - \tau_i(a_k)} \right]^{\hat{a}_i \eta_i}, \]
\[ \eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \]
then Eq. (1.1) is oscillatory.

Proof. To arrive at a contradiction, let us suppose that \( x \) is a nonoscillatory solution of Eq. (1.1). First, we assume that \( x(t) \) is positive for all \( t \in [t_1, \infty)_\tau \), for some \( t_1 \in [t_0, \infty)_\tau \).

Let \( t \in [a_1, b_1]_\tau \), where \( a_1 \geq t_1 \) is sufficiently large. Define
\[ w(t) = -r(t)\frac{\Phi_a(x^3(t))}{\Phi_a(x^\sigma(t))}. \]

It follows that
\[ w^\Delta(t) = \frac{f(t, x)}{\Phi_a(x^\sigma(t))} - \frac{e(t)}{\Phi_a(x^\sigma(t))} + \frac{r(t)\Phi_a(x^3(t))(\Phi_a(x(t)))^\Delta}{\Phi_a(x(t))\Phi_a(x^\sigma(t))}, \tag{3.5} \]
where
\[ f(t, x) = p_0(t) \Phi_a(x(t_0)) + \sum_{i=1}^{n} p_i(t) \Phi_a(x(t_i)). \] (3.6)

Fix \( j, j = 0, 1, 2, \ldots, n \). Clearly, the conditions of Lemma 2.4 are satisfied with \( \tau \) replaced by \( \tau_j \). Therefore, by inequality (2.5) we have
\[ \frac{x(t_j)}{x^a(t)} \geq \frac{\tau_j - \tau_j(a_1)}{\sigma(t)} + \tau_j(a_1), \quad t \in [a_1, b_1]. \] (3.7)

In view of (3.1), (3.7), and the fact that \( \Phi_a \) is increasing, we obtain from (3.6) that
\[ f(t, x) \geq p_0(t) \sum_{i=1}^{n} p_i(t) \frac{t_i(t) - t_i(a_1)}{\sigma(t) - t_i(a_1)} \Phi_a(x^a(t)). \] (3.8)

Using (3.8) in (3.5) we have
\[ w^\Delta(t) \geq p_0(t) \sum_{i=1}^{n} p_i(t) \frac{t_i(t) - t_i(a_1)}{\sigma(t) - t_i(a_1)} + \sum_{i=0}^{n} \eta_i u_i(t) + \frac{r(t)}{\Phi_a(x^a(t))} \frac{\Phi_a(x^a(t))}{\Phi_a(\Phi_a(x^a(t)))^a}. \] (3.9)

where
\[ u_i(t) = \frac{1}{\eta_i} p_i(t) \frac{t_i(t) - t_i(a_1)}{\sigma(t) - t_i(a_1)} \Phi_{\beta_i}(x^a(t)), \quad i \neq 0, \quad u_0(t) = \frac{1}{\eta_0} \Phi_a(x^a(t)). \]

Employing in (3.9) the arithmetic–geometric mean inequality [35],
\[ \sum_{i=0}^{n} \eta_i u_i(t) \geq \prod_{i=0}^{n} u_i^{\eta_i}, \]
we see that
\[ w^\Delta(t) \geq p(t) + \frac{r(t)}{\Phi_a(x^a(t))} \frac{\Phi_a(x^a(t))}{\Phi_a(\Phi_a(x^a(t)))^a}, \] (3.10)

where
\[ p(t) = p_0(t) \sum_{i=1}^{n} p_i(t) \frac{t_i(t) - t_i(a_1)}{\sigma(t) - t_i(a_1)} + \eta e(t) \prod_{i=1}^{n} p_i(t)^{\eta_i} \frac{t_i(t) - t_i(a_1)}{\sigma(t) - t_i(a_1)} \beta_i \eta_i. \]

Multiplying both sides of inequality (3.10) by \( |u^a|^{a+1} \) and then using the identity
\[ (u \Phi_a(u) w)^a = u^a \Phi_a(u^a)w^a + (|u|^{a+1})^a w \]
result in
\[ (u \Phi_a(u) w)^a \geq |u^a|^{a+1} p - |u^a|^{a+1} r + G(u, w), \] (3.11)

where
\[ G(u, w) = |u^a|^{a+1} r + (|u|^{a+1})^a w + |u^a|^{a+1} r \Phi_a(x^a)(\Phi_a(x))^a \Phi_a(\Phi_a(x^a)). \]

We know that [4,7,10,14] \( G(u, w) \geq 0 \), and \( G(u, w) = 0 \) if and only if
\[ u^a = \Phi^{-1}_a(-w/r)u, \] (3.12)

where \( \Phi^{-1}_a \) stands for the inverse function. Since
\[ 1 + \mu \Phi^{-1}_a(-w/r) = x^a/x > 0 \quad \text{and} \quad u(a_1) = 0, \]
Eq. (3.12) has the unique solution \( u \equiv 0 \). Therefore, \( G(u, w) > 0 \) on \( [a_1, b_1] \). We note that in the proof of \( G(u, w) \geq 0 \), Young’s inequality is employed when \( t \) is right-dense, and elementary differential calculus is used when \( t \) is right-scattered; see [4,10].

Now integrating inequality (3.11) from \( a_1 \) to \( b_1 \) and using \( G(u, w) > 0 \) on \( [a_1, b_1] \), we obtain
\[ \int_{a_1}^{b_1} \left\{ |u^a(t)|^{a+1} p(t) - |u^a(t)|^{a+1} r(t) \right\} \Delta t < 0, \]
which of course contradicts (3.4). This completes the proof when \( x(t) \) is eventually positive. The proof when \( x(t) \) is eventually negative is analogous by repeating the arguments on the interval \( [a_2, b_2]_T \) instead of \( [a_1, b_1]_T \). \( \square \)

In Theorem 3.1 one cannot allow \( e(t) \equiv 0 \). In that case we have the next theorem.

**Theorem 3.2.** Suppose that for any given \( T \in [0, \infty)_T \) there exists a subinterval \([a_1, b_1]_T \) of \([T, \infty)_T \) such that

\[
p_i(t) \geq 0 \quad \text{for } t \in [\hat{a}_1, b_1]_T, \quad (i = 0, 1, 2, \ldots, n),
\]

where \( \hat{a}_1 \) is as in (3.3). Let \( \{\eta_i\}, i = 1, 2, \ldots, n \), be an \( n \)-tuple satisfying (2.2) in Lemma 2.2. If there exists a function \( u \in \mathcal{A}(a_1, b_1) \) such that

\[
\int_{a_1}^{b_1} \left\{ |u^\alpha(t)|^{\alpha+1} \tilde{p}(t) - |u^\alpha(t)|^{\alpha+1} r(t) \right\} \Delta t \geq 0,
\]

where

\[
\tilde{p}(t) = p_0(t) \left[ \frac{r_0(t) - r_0(a_1)}{\sigma(t) - r_0(a_1)} \right]^\alpha + \eta \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{r_i(t) - r_i(a_1)}{\sigma(t) - r_i(a_1)} \right]^\beta \eta_i,
\]

then Eq. (1.2) is oscillatory.

**Proof.** We proceed as in Theorem 3.1 to arrive at

\[
w^\alpha(t) \geq p_0(t) \left\{ \frac{r_0(t) - r_0(a_1)}{\sigma(t) - r_0(a_1)} \right\}^\alpha + \sum_{i=1}^{n} \eta_i u_i(t) + \frac{r(t) \Phi_\alpha(x(t)) \Phi_\alpha(x^\alpha(t))^\alpha}{\Phi_\alpha(x(t)) \Phi_\alpha(x^\alpha(t))},
\]

where

\[
u_i(t) = \frac{1}{\eta_i} p_i(t) \left[ \frac{r_i(t) - r_i(a_1)}{\sigma(t) - r_i(a_1)} \right]^\beta \Phi_\beta \alpha(x^\alpha(t)).
\]

Using again the arithmetic–geometric mean inequality

\[
\sum_{i=1}^{n} \eta_i u_i(t) \geq \prod_{i=1}^{n} u_i^{\eta_i}
\]

we have

\[
w^\alpha(t) \geq \tilde{p}(t) + \frac{r(t) \Phi_\alpha(x^\alpha(t)) \Phi_\alpha(x(t))^{\alpha}}{\Phi_\alpha(x(t)) \Phi_\alpha(x^\alpha(t))}.
\]

The remainder of the proof is the same as that of Theorem 3.1. \( \square \)

As shown in [21] for the sublinear terms case we can also remove the sign condition imposed on the coefficients of the sub-half-linear terms to obtain interval criteria which are applicable for the case when some or all of the functions \( q_i(t), i = m + 1, \ldots, n \), are nonpositive. However, we need to assume that the corresponding terms are delay free. More precisely, we consider

\[
\left( r(t) \Phi_\alpha(x^\alpha(t)) \right)^\alpha + p_0(t) \Phi_\alpha(x(t)) + g(t, x) = e(t), \quad t \geq t_0
\]

where

\[
g(t, x) = \sum_{i=1}^{m} p_i(t) \Phi_\beta \alpha(x(t_i)) + \sum_{i=m+1}^{n} p_i(t) \Phi_\beta \alpha(x^\alpha(t)).
\]

It should be noted that the sign condition on the coefficients of the super-half-linear terms cannot be removed alternatively by the same approach. It should also be noted that the function \( e(t) \) cannot vanish on the intervals of interest. The result is as follows.

**Theorem 3.3.** Suppose that for any given \( T \in [0, \infty)_T \) there exist subintervals \([a_1, b_1]_T \) and \([a_2, b_2]_T \) of \([T, \infty)_T \) such that

\[
p_i(t) \geq 0 \quad \text{for } t \in [\hat{a}_1, b_1]_T \cup [\hat{a}_2, b_2]_T, \quad (i = 0, 1, 2, \ldots, m)
\]

and

\[
(-1)^k e(t) > 0 \quad \text{for } t \in [\hat{a}_k, b_k]_T, \quad (k = 1, 2),
\]

such that

\[
p_i(t) \geq 0 \quad \text{for } t \in [\hat{a}_1, b_1]_T \cup [\hat{a}_2, b_2]_T, \quad (i = 0, 1, 2, \ldots, m)
\]

and

\[
(-1)^k e(t) > 0 \quad \text{for } t \in [\hat{a}_k, b_k]_T, \quad (k = 1, 2),
\]

where

\[
\tilde{p}(t) = p_0(t) \left[ \frac{r_0(t) - r_0(a_1)}{\sigma(t) - r_0(a_1)} \right]^\alpha + \eta \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{r_i(t) - r_i(a_1)}{\sigma(t) - r_i(a_1)} \right]^\beta \eta_i.
\]
where
\[ \hat{a}_k = \min \{ \tau_i(a_k) : i = 0, 1, 2, \ldots, m \}. \quad (3.18) \]

If there exist a function \( u \in \mathcal{A}(a_k, b_k) \), \((k = 1, 2)\), and positive numbers \( \lambda_i \) and \( \mu_i \) with
\[ \sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1 \]
such that
\[ \int_{a_k}^{b_k} \left\{ |u^\alpha(t)|^{\alpha+1} \tilde{p}(t) - |u^\alpha(t)|^{\alpha+1} r(t) \right\} \Delta t \geq 0 \quad (3.19) \]
for \( k = 1, 2 \), where
\[ \tilde{p}(t) = \frac{\tau_0(t) - \tau_0(a_1)}{\sigma(t) - \tau_0(a_1)} \alpha + \sum_{i=1}^{m} \frac{\tau_i(t) - \tau_i(a_1)}{\sigma(t) - \tau_i(a_1)} \beta_i - \sum_{i=m+1}^{n} R_i(t), \]
\[ p_i(t) = \beta_i \beta_i - \alpha \beta_i - \alpha \alpha - \beta_i - \alpha \beta_i - \beta_i, \]
\[ R_i(t) = \beta_i (\alpha - \beta_i) \beta_i - \alpha - \beta_i - \alpha - \beta_i, \]
with
\[ (-p_i)^+ (t) = \max \{-p_i(t), 0\}. \]
then Eq. (3.15) is oscillatory.

**Proof.** Suppose that Eq. (1.1) has a nonoscillatory solution \( x \). We may assume that \( x(t) \) is eventually positive on \([a_1, b_1]_\tau\) when \( a_1 \) is sufficiently large. If \( x(t) \) is eventually negative, then one can repeat the proof on the interval \([a_2, b_2]_\tau\).

Clearly,
\[ g(t, x) = \sum_{i=1}^{m} \left[ p_i(t) \Phi_{\beta_i} (x(\tau_i(t))) + \lambda_i |e(t)| \right] - \sum_{i=m+1}^{n} \left[ -p_i(t) x^{\beta_i} \Phi_{\beta_i} (x^{\alpha}(t)) - \mu_i |e(t)| \right] \]
gives
\[ g(t, x) \geq \sum_{i=1}^{m} \left[ p_i(t) \Phi_{\beta_i} (x(\tau_i(t))) + \lambda_i |e(t)| \right] - \sum_{i=m+1}^{n} \left[ (-p_i)^+ (t) \Phi_{\beta_i} (x^{\alpha}(t)) - \mu_i |e(t)| \right]. \]
Applying (2.3) and (2.4) to each summation on the right side with
\[ A = p_i(t), \quad B = \lambda_i |e(t)|, \quad \gamma = \beta_i, \quad \delta = \alpha \quad (\beta_i > \alpha) \]
and
\[ C = (-p_i)^+ (t), \quad D = \mu_i |e(t)|, \quad \delta = \beta_i, \quad \gamma = \alpha \quad (\beta_i < \alpha), \]
we see that
\[ g(t, x) \geq \sum_{i=1}^{m} p_i(t) \Phi_{\alpha} (x(\tau_i(t))) - \sum_{i=m+1}^{n} R_i(t) \Phi_{\alpha} (x^{\alpha}(t)). \quad (3.20) \]
From Eq. (3.15) and inequality (3.20) we have
\[ \left( r(t) \Phi_{\alpha} (x^{\alpha}) \right)^\Delta + p_0(t) \Phi_{\alpha} (x(\tau_0(t))) + \sum_{i=1}^{m} p_i(t) \Phi_{\alpha} (x(\tau_i(t))) - \sum_{i=m+1}^{n} R_i(t) \Phi_{\alpha} (x^{\alpha}(t)) \leq 0. \]
Set
\[ w(t) = -r(t) \frac{\Phi_{\alpha} (x^{\alpha}(t))}{\Phi_{\alpha} (x(t))}. \]
It follows that
\[ w^{\Delta}(t) \geq Q(t) + \frac{r(t) \Phi(x^{\alpha}(t))(\Phi_{\alpha}(x(t)))^\Delta}{\Phi_{\alpha}(x(t)) \Phi_{\alpha}(x^{\alpha}(t))}, \]
where
\[ Q(t) = p_0(t) \frac{\Phi_{\sigma}(x(\tau_0(t)))}{\Phi_{\sigma}(x^\sigma(t))} + \sum_{i=1}^{m} p_i(t) \frac{\Phi_{\sigma}(x(\tau_i(t)))}{\Phi_{\sigma}(x^\sigma(t))} - \sum_{i=m+1}^{n} R_i(t). \]

Now, making use of (3.7), we get the estimate
\[ Q(t) \geq p_0(t) \left[ \frac{\tau_i(t) - \tau_0(a_1)}{\sigma(t) - \tau_0(a_1)} \right]^\alpha + \sum_{i=1}^{m} p_i(t) \left[ \frac{\tau_i(t) - \tau_i(a_1)}{\sigma(t) - \tau_i(a_1)} \right]^\alpha - \sum_{i=m+1}^{n} R_i(t). \]

The remainder of the proof is the same as that of Theorem 3.1, hence omitted. \( \square \)

**Remark 3.4.** Theorems 3.1–3.3 hold for more general equations of the form
\[ (r(t)\Phi_{\sigma}(x^\alpha(t)))^\Delta + p_0(t)g(x(\tau_0(t))) + \sum_{i=1}^{n} p_i(t)f_i(x(\tau_i(t))) = e(t), \quad t \geq t_0 \]
where \( g, f_i : \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions satisfying the growth conditions
\[ xg(x) \geq |x|^\alpha+1 \quad \text{and} \quad xf_i(x) \geq |x|^\beta+1 \quad \text{for all} \ x \in \mathbb{R}. \]

To see this we note that if \( x(t) \) is eventually positive, then taking into account the intervals where the functions \( p_i \) are nonnegative, the above inequalities result in
\[ (r(t)\Phi_{\sigma}(x^\alpha(t)))^\Delta + p_0(t)\Phi_{\sigma}(x(\tau_0(t))) + \sum_{i=1}^{n} p_i(t)\Phi_{\sigma}(x(\tau_i(t))) \leq e(t) \]
for \( t \) sufficiently large. The arguments hereafter follow analogously.

**4. Forms related to Eq. (1.1)**

Related to (1.1) are the dynamic equations with mixed delta and nabla derivatives
\[ (r(t)\Phi_{\sigma}(x^\Delta(t)))^\nabla + p_0(t)\Phi_{\sigma}(x(\tau_0(t))) + \sum_{i=1}^{n} p_i(t)\Phi_{\sigma}(x(\tau_i(t))) = e(t), \] (4.1)
\[ (r(t)\Phi_{\sigma}(x^\nabla(t)))^\Delta + p_0(t)\Phi_{\sigma}(x(\tau_0(t))) + \sum_{i=1}^{n} p_i(t)\Phi_{\sigma}(x(\tau_i(t))) = e(t), \] (4.2)
and
\[ (r(t)\Phi_{\sigma}(x^\nabla(t)))^\nabla + p_0(t)\Phi_{\sigma}(x(\tau_0(t))) + \sum_{i=1}^{n} p_i(t)\Phi_{\sigma}(x(\tau_i(t))) = e(t). \] (4.3)

It is not difficult to see that time scale modifications of the previous arguments give rise to completely parallel results for the above dynamic equations. For an illustrative example we provide below the version of Theorem 3.1 for Eq. (4.1), for which a lemma analogous to Lemma 2.4 can be stated easily. The other theorems for Eqs. (4.1), (4.2), and (4.3) can be obtained by employing arguments developed for Eq. (1.1).

**Theorem 4.1.** Suppose that for any given \( T \in [t_0, \infty)_T \) there exist subintervals \([a_1, b_1]_T\) and \([a_2, b_2]_T\) of \([T, \infty)_T\) such that
\[ p_i(t) \geq 0 \quad \text{for} \ t \in [\hat{a}_i, b_1]_T \cup [\hat{a}_2, b_2]_T, \quad (i = 0, 1, 2, \ldots, n) \]
and
\[ (-1)^k e(t) \geq 0 \quad \text{for} \ t \in [\hat{a}_k, b_k]_T, \quad (k = 1, 2), \]
where \( \hat{a}_k = \min\{\tau_i(a_k) : i = 0, 1, 2, \ldots, n\} \).

Let \( \{\eta_i\}, i = 1, 2, \ldots, n, \) be an n-tuple satisfying (2.1) in Lemma 2.1. If there exists a function \( u \in \mathcal{A}_n(a_k, b_k) := \{ u \in C^1_{[a_k, b_k]} : u(a) = 0 = u(b), u \not\equiv 0 \}, \quad (k = 1, 2), \)
such that
\[ \int_{a_k}^{b_k} \left( |u(t)|^{\alpha+1} p(t) - |u^\nabla(t)|^{\alpha+1} r^\nabla(t) \right) \nabla t \geq 0 \]
for \( k = 1, 2 \), where \( \rho \) denotes the backward jump operator and

\[
p(t) = p_0(t) \left[ \frac{\tau_0(t) - \tau_0(a_k)}{t - \tau_0(a_k)} \right] + \eta|e(t)|^{\eta_0} \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{\tau_i(t) - \tau_i(a_k)}{t - \tau_i(a_k)} \right],
\]

\[
\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=0}^{n} \eta_i^{-\eta_i},
\]

then Eq. (4.1) is oscillatory.

5. Applications

To illustrate the usefulness of the results we state the corresponding theorems in the previous section for the special cases \( \mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}, \) and \( \mathbb{T} = q^\mathbb{N}, (q > 1) \). It is not difficult to provide similar results for other specific time scales of interest.

5.1. Differential equations

If \( \mathbb{T} = \mathbb{R} \), then we have \( f^\Delta = f', \sigma(t) = t, \) and

\[
(r(t)\Phi_0(x'))' + p_0(t)\Phi_0(x(t)) + \sum_{i=1}^{n} p_i(t)\Phi_{\beta_i}(x(\tau_i(t))) = e(t), \quad t \geq t_0,
\]

where \( r, q, q_1, e : [t_0, \infty) \rightarrow \mathbb{R} \) are continuous with \( r > 0 \) nondecreasing; \( \tau_k : [t_0, \infty) \rightarrow \mathbb{R} \) are nondecreasing and continuous with \( \tau_k(t) \leq t, \) \( \lim_{t \to \infty} \tau_k(t) = \infty; \) \( \beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0. \) Let \( \mathcal{A}_1(a, b) := \{ u \in C^1[a, b] : u(a) = 0 = u(b), u \not\equiv 0 \}. \)

The theorems below are extensions of those given by Sun and Wong [21] for \( \alpha = 1 \) to delay differential equations with several delays. Therefore, our results further generalize and extend most of the related literature cited in [21].

**Theorem 5.1.** Suppose that for any given \( T \geq t_0 \) there exist subintervals \( [a_1, b_1] \) and \( [a_2, b_2] \) of \( [T, \infty) \) such that

\[
p_i(t) \geq 0 \quad \text{for} \quad t \in [\hat{a}_1, b_1] \cup [\hat{a}_2, b_2], (i = 0, 1, 2, \ldots, n)
\]

and

\[
(-1)^k e(t) \geq 0 (\not\equiv 0) \quad \text{for} \quad t \in [\hat{a}_k, b_k], (k = 1, 2),
\]

where \( \hat{a}_k = \min(\tau_i(a_k)) : i = 0, 1, 2, \ldots, n. \)

Let \( \{\eta_i\}, i = 1, 2, \ldots, n, \) be an \( n \)-tuple satisfying (2.1) in Lemma 2.1. If there exists a function \( u \in \mathcal{A}_1(a_k, b_k), (k = 1, 2), \) such that

\[
\int_{a_k}^{b_k} \left[ |u(t)|^{\alpha+1}p(t) - |u'(t)|^{\alpha+1}r(t) \right] dt \geq 0
\]

for \( k = 1, 2, \)

\[
p(t) = p_0(t) \left[ \frac{\tau_0(t) - \tau_0(a_k)}{t - \tau_0(a_k)} \right] + \eta|e(t)|^{\eta_0} \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{\tau_i(t) - \tau_i(a_k)}{t - \tau_i(a_k)} \right],
\]

\[
\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=0}^{n} \eta_i^{-\eta_i},
\]

then Eq. (5.1) is oscillatory.

**Theorem 5.2.** Suppose that for any given \( T \geq t_0 \) there exists a subinterval \( [a_1, b_1] \) of \( [T, \infty) \) such that

\[
p_i(t) \geq 0 \quad \text{for} \quad t \in [\hat{a}_1, b_1], (i = 1, 2, \ldots, n),
\]

where \( \hat{a}_k = \min(\tau_i(a_k)) : i = 0, 1, 2, \ldots, n. \)

Let \( \{\eta_i\}, i = 1, 2, \ldots, n, \) be an \( n \)-tuple satisfying (2.1) in Lemma 2.1. If there exists a function \( u \in \mathcal{A}_1(a_1, b_1) \) such that

\[
\int_{a_1}^{b_1} \left[ |u(t)|^{\alpha+1}p(t) - |u'(t)|^{\alpha+1}r(t) \right] dt \geq 0
\]

(5.3)
for $k = 1, 2$, where
\[
p(t) = p_0(t) \left[ \frac{\tau_0(t) - \tau_0(a_k)}{t - \tau_0(a_k)} \right]^\alpha + \eta \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{\tau_i(t) - \tau_i(a_k)}{t - \tau_i(a_k)} \right]^\beta_{\eta_i}, \quad \eta = \prod_{i=1}^{n} \eta_i^{-\eta_i},
\]
then Eq. (5.1) with $e(t) = 0$ is oscillatory.

**Theorem 5.3.** Suppose that for any given $T \geq t_0$ there exist subintervals $[a_1, b_1]$ and $[a_2, b_2]$ of $[T, \infty)$ such that
\[
p_i(t) \geq 0 \quad \text{for } t \in [\hat{a}_i, b_i] \cup [\hat{a}_2, b_2], \quad (i = 0, 1, 2, \ldots, m)
\]
and
\[
(-1)^k e(t) > 0 \quad \text{for } t \in [\hat{a}_k, b_k], \quad (k = 1, 2),
\]
where $\hat{a}_k = \min \{ \tau_i(a_k) : i = 0, 1, 2, \ldots, m \}$.

If there exist a function $u \in \mathcal{A}_1(a_k, b_k), (k = 1, 2)$, and positive numbers $\lambda_i$ and $\mu_i$ with
\[
\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1
\]
such that
\[
\int_{a_k}^{b_k} \left\{ |u(t)|^{\alpha + 1} \tilde{p}(t) - |u'(t)|^{\alpha + 1} r(t) \right\} dt \geq 0 \quad (5.4)
\]
for $k = 1, 2$, where
\[
\tilde{p}(t) = p_0(t) \left[ \frac{\tau_0(t) - \tau_0(a_1)}{t - \tau_0(a_1)} \right]^\alpha + \sum_{i=1}^{m} p_i(t) \left[ \frac{\tau_i(t) - \tau_i(a_1)}{t - \tau_i(a_1)} \right]^\alpha - \sum_{i=m+1}^{n} R_i(t),
\]
\[
p_i(t) = \beta_i(\beta_i - \alpha a_1^{-\beta_i - 1} - \alpha^{-\beta_i} a_1^{-\alpha a_1^{-\beta_i}} p_i^{\beta_i}(t) |e(t)|^{1-\alpha a_1^{-\beta_i}},
\]
\[
R_i(t) = \beta_i(\beta_i - \alpha a_1^{-\beta_i - 1} - \alpha^{-\beta_i} \mu_i^{-\beta_i} ((-p)^+(t))^{\beta_i} |e(t)|^{1-\alpha a_1^{-\beta_i}},
\]
with
\[
(-p)^+(t) = \max(-p(t), 0),
\]
then Eq. (5.1) is oscillatory.

5.2. Difference equations

If $T = \mathbb{Z}$, then we have $f^A(k) = \Delta f(k) = f(k+1) - f(k)$, $\sigma(k) = k + 1$, and
\[
\Delta (r(k) \phi(x(k))) + p_0(k) \phi(x(\tau_0(k))) + \sum_{i=1}^{n} p_i(k) \phi(x(\tau_i(k))) = e(k), \quad k \geq k_0, \quad (5.5)
\]
where $r, q, e : [k_0, \infty) \rightarrow \mathbb{R}$ are functions with $\Delta r(k) > 0$, and $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$. Let $[a, b] = [a, a + 1, a + 2, \ldots, b]$, and $\mathcal{A}_2(a, b) := \{ u : [a, b] \rightarrow \mathbb{R}, u(a) = 0 = u(b), u \neq 0 \}$.

**Theorem 5.4.** Suppose that for any given $K \geq k_0$ there exist subintervals $[a_1, b_1]_{\mathbb{N}}$ and $[a_2, b_2]_{\mathbb{N}}$ of $[K, \infty)_{\mathbb{N}}$ such that
\[
p_i(j) \geq 0 \quad \text{for } j \in [\hat{a}_1, b_1]_{\mathbb{N}} \cup [\hat{a}_2, b_2]_{\mathbb{N}}, \quad (i = 0, 1, 2, \ldots, n)
\]
and
\[
(-1)^k e(j) \geq 0 \quad (\neq 0) \quad \text{for } j \in [\hat{a}_k, b_k]_{\mathbb{N}}, \quad (k = 1, 2),
\]
where $\hat{a}_k = \min \{ \tau_i(a_k) : i = 0, 1, 2, \ldots, n \}$.

Let $\{ \eta_i \}, i = 1, 2, \ldots, n$, be an $n$-tuple satisfying (2.1) in Lemma 2.1. If there exists a function $u \in \mathcal{A}_2(a_k, b_k), (k = 1, 2)$, such that
\[
\sum_{j=a_k}^{b_k-1} \left\{ |u(j + 1)|^{\alpha + 1} p(j) - |\Delta u(j)|^{\alpha + 1} r(j) \right\} \geq 0 \quad (5.6)
\]
for $k = 1, 2$, where

$$p(j) = p_0(j) \sum_{i=1}^{n} p_i(j) \eta_i \left( \frac{\tau_i(j) - \tau_i(a_k)}{j + 1 - \tau_i(a_k)} \right)^{\alpha_i} + \eta \epsilon(j) \left( \frac{\tau_i(j) - \tau_i(a_k)}{j + 1 - \tau_i(a_k)} \right)^{\beta_i},$$

$$\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=1}^{n} \eta_i^{-\eta_i}.$$

then Eq. (5.5) is oscillatory.

**Theorem 5.5.** Suppose that for any given $K \geq k_0$ there exists a subinterval $[a_1, b_1]$ such that

$$p_i(j) \geq 0 \quad \text{for } j \in [\hat{a}_1, b_1], \quad (i = 0, 1, 2, \ldots, n),$$

where $\hat{a}_k = \min\{\tau_i(a_k) : i = 0, 1, 2, \ldots, n\}$.

Let $\eta_i, i = 1, 2, \ldots, n$, be an $n$-tuple satisfying (2.2) in Lemma 2.2. If there exists a function $u \in A_2(a_1, b_1)$ such that

$$\sum_{j=1}^{b_k-1} \left| \frac{u(j+1)}{u(j)} \right|^{\nu+1} p(j) - |\Delta u(j)|^{\nu+1} \epsilon(j) \geq 0 \tag{5.7}$$

for $k = 1, 2$, where

$$p(j) = p_0(j) \sum_{i=1}^{n} p_i(j) \eta_i \left( \frac{\tau_i(j) - \tau_i(a_k)}{j + 1 - \tau_i(a_k)} \right)^{\alpha_i} + \eta \epsilon(j) \left( \frac{\tau_i(j) - \tau_i(a_k)}{j + 1 - \tau_i(a_k)} \right)^{\beta_i}, \quad \eta = \prod_{i=1}^{n} \eta_i^{-\eta_i}.$$

then Eq. (5.5) with $\epsilon(k) \equiv 0$ is oscillatory.

**Theorem 5.6.** Suppose that for any given $K \geq k_0$ there exists subintervals $[a_1, b_1]$ and $[a_2, b_2]$ such that

$$p_i(j) \geq 0 \quad \text{for } j \in [\hat{a}_1, b_1] \cup [\hat{a}_2, b_2], \quad (i = 0, 1, 2, \ldots, m)$$

and

$$(-1)^k \epsilon(j) > 0 \quad \text{for } j \in [\hat{a}_1, b_1], \quad (k = 1, 2),$$

where $\hat{a}_k = \min\{\tau_i(a_k) : i = 0, 1, 2, \ldots, m\}$.

If there exist a function $u \in A_2(a_k, b_k), (k = 1, 2)$, and positive numbers $\lambda_i$ and $\mu_i$ with

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1$$

such that

$$\sum_{j=\hat{a}_k}^{b_k-1} \left| \frac{u(j+1)}{u(j)} \right|^{\nu+1} \tilde{p}(j) - |\Delta u(j)|^{\nu+1} \epsilon(j) \geq 0 \tag{5.8}$$

for $k = 1, 2$, where

$$\tilde{p}(j) = p_0(j) \sum_{i=1}^{n} P_i(j) \left( \frac{\tau_i(j) - \tau_i(a_k)}{j + 1 - \tau_i(a_k)} \right)^{\alpha_i} + \sum_{i=m+1}^{n} R_i(j),$$

$$P_i(j) = \beta_i \left( \frac{\beta_i - \alpha}{\beta_i - \beta_i} \right)^{\alpha - \beta_i} \lambda_i^{1 - \alpha/\beta_i} \mu_i^{1 - \alpha/\beta_i} |\epsilon(j)|^{1 - \alpha/\beta_i},$$

$$R_i(j) = \beta_i \left( \frac{\beta_i - \alpha}{\beta_i - \beta_i} \right)^{\alpha - \beta_i} \mu_i^{1 - \alpha/\beta_i} |(-p_i)^+(t)| \epsilon(t)^{1 - \alpha/\beta_i}$$

with

$$(-p_i)^+(j) = \max\{-p_i(j), 0\},$$

then Eq. (5.5) is oscillatory.
5.3. $q$-Difference equations

If $T = q^N$ with $q > 1$, then $\sigma(t) = qt, f^A(t) = \Delta_q f(t) \equiv \Delta_q f(t) = (f(qt) - f(t))/(qt - t)$, and

$$\Delta_q \{r(t) \Phi_\alpha \Delta_q x(t)\} + p(t) \Phi_\alpha (\lambda(t)) + \sum_{i=1}^{n} p_i(t) \Phi_{\beta_1}(\lambda_i(t)) = e(t), \quad t \in [t_0, \infty)_q,$$

(5.9)

where $[t_0, \infty)_q = \{q^0, q^{0+1}, q^{0+2}, \ldots\}$ denotes a $q$-half-line, $r, p, p_i, e : [t_0, \infty)_q \to \mathbb{R}$ with $r(qt) > r(t) > 0$, and $\beta_1 > \cdots > \beta_m > \alpha > \beta_{m+1} > \cdots > \beta_n > 0$. Let $[a, b]_q = \{q^0, q^{0+1}, q^{0+2}, \ldots, q^b\}$, and $\mathcal{A}_3(a, b) : = \{u : [a, b]_q \to \mathbb{R}, u(a) = 0 = u(b), u \neq 0\}$.

**Theorem 5.7.** Suppose that for any given $T \in [t_0, \infty)_q$ there exist subintervals $[a_i, b_1]_q$ and $[a_1, b_2]_q$ of $[T, \infty)_q$ such that

$$p_i(t) \geq 0 \quad \text{for} \quad t \in [\hat{a}_i, b_1]_q \cup [\hat{a}_2, b_2]_q, (i = 0, 1, 2, \ldots, n)$$

and

$$(-1)^k e(t) \geq 0 (\neq 0) \quad \text{for} \quad t \in [\hat{a}_k, b_k]_q, (k = 1, 2),$$

where $\hat{a}_k = \min\{\tau_i(a_k) : i = 0, 1, 2, \ldots, n\}$.

Let $\{\eta_i\}, i = 1, 2, \ldots, n$, be an $n$-tuple satisfying (2.1) in Lemma 2.1. If there exists a function $u \in \mathcal{A}_3(a_k, b_k), (k = 1, 2)$, such that

$$\sum_{j=\hat{a}_k}^{b_k-1} \{q^i u(q^{i+1})(\alpha + 1) p(q^i) - |\Delta_q u(q^i)|^{\alpha + 1} r(q^i) \} \geq 0$$

(5.10)

for $k = 1, 2$, where

$$p(t) = p_0(t) \left[ \frac{\tau_0(t) - \tau_0(a_k)}{qt - \tau_0(a_k)} \right]^q + \eta |e(t)|^n \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{\tau_i(t) - \tau_i(a_k)}{qt - \tau_i(a_k)} \right]^{\beta_i \eta_i},$$

$$\eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=1}^{n} \eta_i^{-\eta_i},$$

then Eq. (5.9) is oscillatory.

**Theorem 5.8.** Suppose that for any given $T \in [t_0, \infty)_q$ there exist a subinterval $[a_i, b_1]_q$ of $[T, \infty)_q$ such that

$$p_i(t) \geq 0 \quad \text{for} \quad t \in [\hat{a}_1, b_1]_q, (i = 0, 1, 2, \ldots, n),$$

where $\hat{a}_k = \min\{\tau_i(a_k) : i = 0, 1, 2, \ldots, n\}$.

Let $\{\eta_i\}, i = 1, 2, \ldots, n$, be an $n$-tuple satisfying (2.2) in Lemma 2.2. If there exists a function $u \in \mathcal{A}_3(a_1, b_1)$ such that

$$\sum_{j=\hat{a}_1}^{b_1-1} \{q^i u(q^{i+1})(\alpha + 1) p(q^i) - |\Delta_q u(q^i)|^{\alpha + 1} r(q^i) \} \geq 0$$

(5.11)

for $k = 1, 2$, where

$$p(t) = p_0(t) \left[ \frac{\tau_0(t) - \tau_0(a_k)}{qt - \tau_0(a_k)} \right]^q + \eta \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{\tau_i(t) - \tau_i(a_k)}{qt - \tau_i(a_k)} \right]^{\beta_i \eta_i}, \quad \eta = \prod_{i=1}^{n} \eta_i^{-\eta_i},$$

then Eq. (5.9) with $e(t) \equiv 0$ is oscillatory.

**Theorem 5.9.** Suppose that for any given $T \in \mathbb{T}$ there exist subintervals $[a_1, b_1]_q$ and $[a_2, b_2]_q$ of $[T, \infty)_q$ such that

$$p_i(t) \geq 0 \quad \text{for} \quad t \in [\hat{a}_1, b_1]_q \cup [\hat{a}_2, b_2]_q, (i = 1, 2, \ldots, m)$$

and

$$(-1)^k e(t) \geq 0 \quad \text{for} \quad t \in [\hat{a}_k, b_k]_q, (k = 1, 2),$$

where $\hat{a}_k = \min\{\tau_i(a_k) : i = 0, 1, 2, \ldots, m\}$.

If there exist a function $u \in \mathcal{A}_3(a_k, b_k), (k = 1, 2)$, and positive numbers $\lambda_i$ and $\mu_i$ with

$$\sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1$$

then
such that
\[ \sum_{j=q}^{b_2-1} q_j \left| u(q_j^+) |^{α+1} P_j(q_j) - |Δq u(q_j^+) |^{α+1} R_j(q_j) \right| ≥ 0 \quad (5.12) \]
for \( k = 1, 2 \), where
\[ P_j(t) = \frac{1}{r(t(t))} \left( \tau(t) - \tau(a_1) \right) + \sum_{i=1}^{m} \frac{1}{r(t_i)} \left( \tau(t_i) - \tau(a_1) \right) - \sum_{i=m+1}^{n} R_i(t), \]
\[ P_i(t) = \beta_i(\alpha - \beta_i)^{α/β_i - 1} \alpha^{-α} \beta_i^{1-α/β_i} \lambda_i^{α/β_i} p_i(t) |e(t)|^{1-α/β_i}, \]
\[ R_i(t) = \beta_i(\alpha - \beta_i)^{α/β_i - 1} \alpha^{-α} \beta_i^{1-α/β_i} \lambda_i^{α/β_i} \left( -p_i^+(t) \right)^{α/β_i} |e(t)|^{1-α/β_i} \]
with
\[ (-p_i)^+(t) = \max\{-p_i(t), 0\}. \]
then Eq. (5.9) is oscillatory.

6. Advance type equations with mixed nonlinearities

We consider the second-order nonlinear dynamic equations
\[ (r(t)Φ_a(x^s))^{Δ} + p_0(t)Φ_a(x(θ(t(t))) + \sum_{i=1}^{n} p_i(t)Φ_{β_i}(x(θ_i(t))) = e(t), \quad t \in [t_0, ∞)_T. \quad (6.1) \]
and
\[ (r(t)Φ_a(x^s))^{Δ} + p_0(t)Φ_a(x(θ(t(t))) + \sum_{i=1}^{n} p_i(t)Φ_{β_i}(x(θ_i(t))) = 0, \quad t \in [t_0, ∞)_T, \quad (6.2) \]
where \( Φ_a(u) = |u|^{α-1} u \); the functions \( r, p_i, e : T → R \) are right-dense continuous with \( r > 0 \) nondecreasing; the advances \( θ_k : [t_0, ∞)_T → T \) are nondecreasing and right-dense continuous with \( θ(k) ≥ t \); and the exponents satisfy
\[ β_1 > \cdots > β_m > α > β_{m+1} > \cdots > β_n > 0. \]

Similar to the arguments in Lemma 2.4 one can easily prove the following lemma; see [5].

**Lemma 6.1.** Let \( θ : T → T \) be a nondecreasing and right-dense continuous function such that \( θ(t) ≥ t \), and \( a, b ∈ T \) with \( a < b \). If \( x ∈ C^1_0([a, θ(b)])_T \) is a positive function for which \( r(t)Φ_a(x^s(t)) \) is nonincreasing on \([a, θ(b)]_T\), then
\[ \frac{x(θ(t))}{x^s(t)} ≥ \frac{θ(b) - θ(t)}{θ(b) - σ(t)}, \quad t \in [a, b]_T. \quad (6.3) \]

We now state the following theorems.

**Theorem 6.2.** Suppose that for any given \( T ∈ [t_0, ∞)_T \) there exist subintervals \([a_1, b_1]_T\) and \([a_2, b_2]_T\) of \([T, ∞)_T\) such that
\[ p_i(t) ≥ 0 \quad \forall t ∈ [a_1, b_1]_T \cup [a_2, b_2]_T, (i = 0, 1, 2, \ldots, n) \quad (6.4) \]
and
\[ (-1)^k e(t) ≥ 0 (≠ 0) \quad \forall t ∈ [a_k, b_k]_T, (k = 1, 2), \]
where
\[ a_k = \min\{θ_i(a_k) : i = 0, 1, 2, \ldots, n\}. \quad (6.5) \]
Let \( \{η_i\}, i = 1, 2, \ldots, n \) be an \( n \)-tuple satisfying (2.1) in Lemma 2.1. If there exists a function \( u ∈ A(a_k, b_k), (k = 1, 2) \), such that
\[ \int_{a_k}^{b_k} \left| u^s(t) |^{α+1} p(t) - |u^s(t) |^{α+1} r(t) \right| Δt ≥ 0 \]
for $k = 1, 2$, where
\[ p(t) = p_0(t) \left[ \frac{\theta_0(b_k) - \theta_0(t)}{\theta_0(b_k) - \sigma(t)} \right]^\alpha + \eta |e(t)|^{\eta_0} \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{\theta_i(b_k) - \theta_i(t)}{\theta_i(b_k) - \sigma(t)} \right]^{\frac{\hat{R}}{\eta_i}}, \]
\[ \eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \quad \eta = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \]
then Eq. (6.1) is oscillatory.

**Theorem 6.3.** Suppose that for any given $T \in [t_0, \infty)$ there exists a subinterval $[a_1, b_1]$ of $[T, \infty)$ such that
\[ p_i(t) \geq 0 \quad \text{for} \ t \in [\hat{a}_1, b_1], \quad (i = 0, 1, 2, \ldots, n), \]
where $\hat{a}_i$ as in (6.5).
Let $\{\eta_i\}, i = 1, 2, \ldots, n$, be an $n$-tuple satisfying (2.2) in Lemma 2.2. If there exists a function $u \in \mathcal{A}(a_1, b_1)$ such that
\[ \int_{a_1}^{b_1} \left\{ |u^\alpha(t)|^{\alpha+1} p(t) - |u^\Delta(t)|^{\alpha+1} r(t) \right\} \Delta t \geq 0, \]
where
\[ \tilde{p}(t) = p_0(t) \left[ \frac{\theta_0(b_1) - \theta_0(t)}{\theta_0(b_1) - \sigma(t)} \right]^\alpha + \eta \prod_{i=1}^{n} p_i(t)^{\eta_i} \left[ \frac{\theta_i(b_1) - \theta_i(t)}{\theta_i(b_1) - \sigma(t)} \right]^{\frac{\hat{R}}{\eta_i}}, \quad \eta = \prod_{i=1}^{n} \eta_i^{-\eta_i}, \]
then Eq. (6.2) is oscillatory.

The next result is concerned with the equation
\[ (r(t) \Phi_a(x^a))^{\lambda} + p_0(t) \Phi_a(x(\theta_0(t))) + g(t, x) = e(t), \quad t \in [t_0, \infty) \]
where
\[ g(t, x) = \sum_{i=1}^{m} p_i(t) \Phi_{\beta_i}(x(\theta_i(t))) + \sum_{i=m+1}^{n} p_i(t) \Phi_{\beta_i}(x^a(t)). \]

**Theorem 6.4.** Suppose that for any given $T \in [t_0, \infty)$ there exist subintervals $[a_1, b_1]$ and $[a_2, b_2]$ of $[T, \infty)$ such that
\[ p_i(t) \geq 0 \quad \text{for} \ t \in [\hat{a}_1, b_1] \cup [\hat{a}_2, b_2], \quad (i = 0, 1, 2, \ldots, m) \]
and
\[ (-1)^k e(t) > 0 \quad \text{for} \ t \in [\hat{a}_k, b_k], \quad (k = 1, 2), \]
where
\[ \hat{a}_k = \min\{\theta_i(a_k) : i = 0, 1, 2, \ldots, m\}. \]
If there exist a function $u \in \mathcal{A}(a_k, b_k)$, $(k = 1, 2)$, and positive numbers $\lambda_i$ and $\mu_i$ with
\[ \sum_{i=1}^{m} \lambda_i + \sum_{i=m+1}^{n} \mu_i = 1 \]
such that
\[ \int_{a_k}^{b_k} \left\{ |u^\alpha(t)|^{\alpha+1} \tilde{p}(t) - |u^\Delta(t)|^{\alpha+1} r(t) \right\} \Delta t \geq 0 \]
for $k = 1, 2$, where
\[ \tilde{p}(t) = p_0(t) \left[ \frac{\theta_0(b_k) - \theta_0(t)}{\theta_0(b_k) - \sigma(t)} \right]^\alpha + \sum_{i=1}^{m} p_i(t) \left[ \frac{\theta_i(b_k) - \theta_i(t)}{\theta_i(b_k) - \sigma(t)} \right]^{\alpha} - \sum_{i=m+1}^{n} R_i(t), \]
\[ p_i(t) = \beta_i (\alpha - \alpha)^{\alpha/\beta_i - 1} - \alpha/\beta_i \lambda_i \mu_i^{1-\alpha/\beta_i} r(t)^{1-\alpha/\beta_i}, \]
\[ R_i(t) = \beta_i (\alpha - \alpha)^{\alpha/\beta_i - 1} - \alpha/\beta_i \mu_i^{1-\alpha/\beta_i} \left| (-p_i)^\alpha (t)^{1-\alpha/\beta_i} r(t) \right|^{1-\alpha/\beta_i} \]
with
\[ (-p_i)^\alpha (t) = \max\{-p_i(t), 0\}, \]
then Eq. (6.6) is oscillatory.
Remark 6.5. As in the delay case, theorems in this section remain valid for equations of the form
\[
\left( r(t)\phi_u(x^{\Delta}(t)) \right)^{\Delta} + p_0(t)\phi_u(x(\theta_0(t))) + \sum_{i=1}^{n} p_i(t)f_i(x(\theta_i(t))) = e(t), \quad t \geq t_0
\]
where \( g, f_i : \mathbb{R} \to \mathbb{R} \) are continuous and satisfy
\[
\begin{align*}
 xg(x) &\geq |x|^{\alpha+1} \quad \text{and} \quad xf_i(x) \geq |x|^{\beta_i+1} \quad \text{for all} \quad x \in \mathbb{R}.
\end{align*}
\]

7. Extended delay and advance functions

As pointed out by Čermák [36] the assumptions
\[
\tau : \mathbb{T} \to \mathbb{T}, \quad (\tau(t) \leq t)
\]
and
\[
\theta : \mathbb{T} \to \mathbb{T}, \quad (\theta(t) \geq t),
\]
although quite natural, are rather restrictive especially in the asymptotic investigations of dynamic equations. In [36] the extended backward jump operator
\[
\rho^+(s) = \begin{cases} 
\sup\{u \in \mathbb{T}, \ u \leq s, \ \text{for} \ s > \inf\mathbb{T}\} 
\quad \text{for} \ s > \inf\mathbb{T} \\
\inf\mathbb{T}, \quad \text{for} \ s \leq \inf\mathbb{T} 
\end{cases}
\]
is introduced to study asymptotic behavior of dynamic equations. Following this idea we may also consider a broader class of dynamic equations.

In addition to the extended backward jump operator \( \rho^+ \) we define also the extended forward jump operator \( \sigma^+ \) as follows
\[
\sigma^+(s) = \begin{cases} 
\inf\{u \in \mathbb{T}, \ u \geq s, \ \text{for} \ s < \sup\mathbb{T}\} 
\quad \text{for} \ s < \sup\mathbb{T} \\
\sup\mathbb{T}, \quad \text{for} \ s \geq \sup\mathbb{T}.
\end{cases}
\]

Let \( \tau^+ \) and \( \theta^+ \) denote the functions defined by
\[
\tau^+(t) = \rho^+(\tau(t))
\]
and
\[
\theta^+(t) = \sigma^+(\theta(t)).
\]

Note that \( \tau^+_i(t) \leq \tau_i(t) \) and \( \theta^+_i(t) \geq \theta_i(t) \). In fact,
\[
\tau^+_i(t) = \tau_i(t) \quad \text{if} \ \tau_i(t) \in \mathbb{T} \quad \text{for} \ t \in \mathbb{T}, \quad \text{and} \quad \tau^+_i(t) \leq \tau_i(t) \quad \text{if} \ \tau_i(t) \not\in \mathbb{T},
\]
and
\[
\theta^+_i(t) = \theta_i(t) \quad \text{if} \ \theta_i(t) \in \mathbb{T} \quad \text{for} \ t \in \mathbb{T}, \quad \text{and} \quad \theta^+_i(t) \geq \theta_i(t) \quad \text{if} \ \theta_i(t) \not\in \mathbb{T}.
\]

Therefore we call \( \tau^+ \) and \( \theta^+ \) the extended delay and advance functions, respectively, although we note that since any time scale \( \mathbb{T} \) is topologically closed, \( \tau^+ \) and \( \theta^+ \) may not sent the time scale into itself.

We consider the extended forms of Eqs. (1.1) and (6.1) as follows, namely
\[
\left( r(t)\phi_u(x^{\Delta}(t)) \right)^{\Delta} + p_0(t)\phi_u(x(\theta_0(t))) + \sum_{i=1}^{n} p_i(t)\phi_{\beta_i}(x(\theta_i(t))) = e(t), \quad t \in [t_0, \infty)_{\mathbb{T}} \tag{7.1}
\]
and
\[
\left( r(t)\phi_u(x^{\Delta}(t)) \right)^{\Delta} + p_0(t)\phi_u(x(\theta_0(t))) + \sum_{i=1}^{n} p_i(t)\phi_{\beta_i}(x(\theta_i^+(t))) = e(t), \quad [t_0, \infty)_{\mathbb{T}}. \tag{7.2}
\]

It is not difficult to see that all theorems obtained in this paper are valid for Eqs. (7.1) and (7.2). One needs only to replace \( \tau_i \) by \( \tau^+_i \) and \( \theta_i \) by \( \theta^+_i \) everywhere it applies.

Finally, we note that analogous results can be obtained for equations with mixed nonlinearities having both delay and advance arguments as in [7].
8. Examples

We shall illustrate Theorem 3.2 in the case when $T = \mathbb{R}$ and $T = \mathbb{Z}$, where for both we take $n = 2, m = 1, \alpha = 2, \beta_1 = 3, \beta_2 = 1, r(t) \equiv 1, p_0(t) \equiv A, p_1(t) \equiv B, p_2(t) \equiv C$, with the constant coefficients $A, B, C$ satisfying $A, B, C > 0$. We also take $e(t) \equiv 0$, and have

$$\{\eta_1, \eta_2\} = \left\{ \frac{\alpha - \beta_2}{\beta_1 - \beta_2}, \frac{\beta_1 - \alpha}{\beta_1 - \beta_2} \right\} = \left\{ \frac{1}{2}, \frac{1}{2} \right\}$$

as a 2-tuple satisfying (2.2) in Lemma 2.2.

Example 8.1. Consider on $\mathbb{R}$ the differential equation

$$\left( |\dot{x}(t)| \dot{x}(t) \right)' + A|x(t - 2)|x(t - 2) + B|x(t - 1)|^2 x(t - 1) + Cx(t - 1) = 0, \quad t \in [0, \infty). \tag{8.1}$$

We see that

$$\tau_0(t) = t - 2, \quad \tau_1(t) = \tau_2(t) = t - 1.$$ 

Since $p_0(t) \equiv A, p_1(t) \equiv B, p_2(t) \equiv C$ with $A, B, C > 0$, condition (3.13) in Theorem 3.2 is satisfied. Let $a_1 = j$ and $b_1 = j + \pi$ for integers $j$, and $u(t) = \sin(t - j)$. Then $u \in \mathcal{A}(j, j + \pi)$, and

$$\int_{j}^{j+\pi} \left\{ |u(t)|^2 \tilde{p}(t) - |u'(t)|^2 \right\} dt = -1.33333 + 0.254095A + 0.963843\sqrt{B}\sqrt{C},$$

where

$$\tilde{p}(t) = A \left(\frac{t - j}{t - j + 2}\right)^2 + 2\sqrt{B}\sqrt{C} \left(\frac{t - j}{t - j + 1}\right)^2.$$

Clearly, if the constant coefficients $A, B, C > 0$ satisfy the inequality

$$A + 3.79323\sqrt{B}\sqrt{C} \geq 5.24737,$$

then the above integral is nonnegative and hence Eq. (8.1) is oscillatory by Theorem 3.2.

Example 8.2. Consider on $\mathbb{Z}$ the difference equation

$$\Delta(|\Delta x(t)|\Delta x(t)) + A|x(t - 3)|x(t - 3) + B|x(t - 1)|^2 x(t - 1) + Cx(t - 2) = 0, \quad t \in \{0, 1, 2, \ldots\}. \tag{8.2}$$

We have

$$\tau_0(t) = t - 3, \quad \tau_1(t) = t - 1, \quad \tau_2(t) = t - 2.$$ 

Again condition (3.13) in Theorem 3.2 is satisfied, as is (2.2) in Lemma 2.2. Let $a_1 = 3j$ and $b_1 = 3j + 3$ for integers $j$, and $u(t) = t \mod 3$. Then $u \in \mathcal{A}(3j, 3j + 3)$, and

$$\sum_{t=3j}^{3j+2} \left\{ |u(t + 1)|^2 \tilde{p}(t) - |\Delta u(t)|^3 \right\} = -10 + \frac{8A}{25} + \frac{8\sqrt{B}\sqrt{C}}{3\sqrt{3}},$$

where

$$\tilde{p}(t) = A \left(\frac{t - 3j}{t + 4 - 3j}\right)^2 + 2\sqrt{B}\sqrt{C} \left(\frac{t - 3j}{t + 2 - 3j}\right)^{3/2} \left(\frac{t - 3j}{t + 3 - 3j}\right)^{1/2}.$$

Consequently, if the constant coefficients $A, B, C > 0$ satisfy the relation

$$A + \frac{25}{3} \sqrt{\frac{BC}{3}} \geq \frac{125}{4},$$

then the above sum is nonnegative and hence Eq. (8.2) is oscillatory by Theorem 3.2.
References