Positive solutions to semi-positone second-order three-point problems on time scales

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\textbf{Abstract}

Using a fixed point theorem of generalized cone expansion and compression we establish the existence of at least two positive solutions for the nonlinear semi-positone three-point boundary value problem on time scales

\[ u^{\Delta^2}(t) + f(t, u(t)) = 0, \quad t \in (a, T], \]

\[ u(a) = 0, \quad u(T) = 0. \]

Here \( t \in [a, T], \) where \( T \) is a time scale, \( \alpha > 0, \eta \in (a, \rho(T)), \) \( \alpha(\eta-a) < T-a, \) and the parameter \( \lambda > 0 \) belongs to a certain interval. These results are new for difference equations as well as for general time scales. An example is provided for differential, difference, and \( q-\)difference equations.

\section{1. Introduction to the boundary value problem}

We will be concerned with proving the existence of positive solutions to the semi-positone second-order three-point nonlinear boundary value problem on a time scale \( \mathbb{T} \) given by

\[ u^{\Delta^2}(t) + f(t, u(t)) = 0, \quad t \in (a, T], \]

\[ u(a) = 0, \quad u(T) = 0. \]

where \( \Delta \) is the delta derivative and \( \nabla \) is the nabla derivative. Throughout the paper we assume \( \eta \in (a, \rho(T)), \) for \( a \in \mathbb{T}, \) \( T \in \mathbb{T}^+. \) \( \alpha > 0, \) and \( \alpha(\eta-a) < T-a, \) We likewise assume that \( f : [a, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous, and \( f(t, \cdot) \) does not vanish identically on any subset of \([a, T].\) By a positive solution of (1.1) and (1.2) we understand a function \( u \) which is positive on \([a, T],\) and satisfies dynamic equation (1.1) and boundary conditions (1.2). For more on time scales and the time-scale calculus, please see the book by Bohner and Peterson [7].

Eq. (1.1) is a dynamic equation on time scales, dynamic in the sense that the specific choice of time scale determines the interpretation of the delta and nabla derivatives. For example, we have the following versions of Eq. (1.1):

- \( \mathbb{T} = \mathbb{R} \) (differential equations): \( u''(t) + f(t, u(t)) = 0, \quad t \in (a, T], \)
- \( \mathbb{T} = \mathbb{Z} \) (difference equations): \( \Delta^2 u(t) - 1 + f(t, u(t)) = 0, \quad t \in (a, T], \)
- \( \mathbb{T} = q^\mathbb{Z} \) (quantum equations): \( D^2(D_q u)(t) + f(t, u(t)) = 0, \quad t \in (a, T], \)

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where \( \Delta x(t) = x(t+1) - x(t) \) with \( \Delta^2 x(t) = \Delta(\Delta x(t)) \), while for \( q > 1 \) we have
\[
D^q x(t) = \frac{x(t) - x(t/q)}{t - t/q} \quad \text{and} \quad D_q x(t) = \frac{x(qt) - x(t)}{t(q-1)}.
\]
Boundary value problem (1.1) and (1.2) was studied in the differential equations case recently [15]; the aim of this sequel is to extend these results to difference equations, quantum equations, and arbitrary dynamic equations on time scales. Other recent work on second-order boundary value problems on time scales include [2–5, 8–11]. Earlier papers on second-order three-point problems in the continuous case include [12–14]. The nabla derivative was introduced in [6].

We now present a fixed point theorem of generalized cone expansion and compression which will be used in the later proofs. Let \( E \) be a real Banach space and \( P \) be a cone in \( E \), and let \( \theta \) denote the null element. The map \( \gamma : P \to \mathbb{R}^1 \) is said to be a convex functional on \( P \) provided that \( \gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y) \) for all \( x, y \in P \) and \( t \in [0, 1] \). See [16] for further information.

**Theorem 1.1** (See [16]). Let \( \Omega_1, \Omega_2 \) be two open bounded subsets in \( E \) with \( \theta \in \Omega_1 \) and \( \overline{\Omega}_1 \subset \Omega_2 \). Suppose that \( A : \partial P \cap (\Omega_2 \setminus \Omega_1) \to P \) is completely continuous and \( \gamma : P \to [0, \infty) \) is a uniformly continuous convex functional with \( \gamma(\theta) = 0 \) and \( \gamma(x) > 0 \) for \( x \neq \theta \). If one of the two conditions

(i) \( \gamma(Ax) \leq \gamma(x) \) for all \( x \in P \cap \partial \Omega_1 \) and \( \inf_{x \in P \cap \partial \Omega_2} \gamma(x) > 0 \), \( \gamma(Ax) \geq \gamma(x) \) for all \( x \in P \cap \partial \Omega_2 \), and

(ii) \( \inf_{x \in P \cap \partial \Omega_1} \gamma(x) > 0 \), \( \gamma(Ax) \geq \gamma(x) \) for all \( x \in P \cap \partial \Omega_1 \) and \( \gamma(Ax) \leq \gamma(x) \) for all \( x \in P \cap \partial \Omega_2 \)

is satisfied, then \( A \) has at least one fixed point in \( P \cap (\Omega_2 \setminus \Omega_1) \).

The paper is organized as follows. In Section 2, we give some preliminary results that will be used in the proof of the main result. In Section 3, we prove the existence of at least two positive solutions for problem (1.1) and (1.2). In the end, we illustrate a simple use of the main result.

2. Foundational lemmas

To prove the main existence result we will employ several straightforward lemmas. These lemmas are based on the boundary value problem

\[ u^{\Delta\Delta}(t) + y(t) = 0, \quad t \in (a, T), \]  
\[ u(a) = 0, \quad u(\eta) = u(T). \]  

**Lemma 2.1.** If \( \alpha(\eta - a) \neq T - a \), then for \( y \in C\Delta[a, T], \) the boundary value problem (2.1) and (2.2) has the unique solution

\[ u(t) = -\int_a^t \frac{(t-s)y(s)\Delta s}{1 - \frac{\alpha(\eta - a)}{T - a}} + \int_a^T \frac{(T-s)y(s)\Delta s}{1 - \frac{\alpha(\eta - a)}{T - a}}, \]  

where \( d := T - a - \alpha(\eta - a) \neq 0 \).

**Proof.** Let \( u \) be as in (2.3). Routine calculations verify that \( u \) satisfies the boundary conditions in (2.2). By [7, Theorem 8.50 (iii)],

\[ \left( \int_a^t f(s)\Delta s \right)^{\Delta} = f(\sigma(t), \sigma(t)) + \int_a^t f^\Delta(t)\Delta s \]  

if \( f, f^\Delta \) are continuous. Using this theorem to take the delta derivative of (2.3) we have

\[ u^\Delta(t) = -\int_a^t y(s)\Delta s - \frac{\alpha}{d} \int_a^\eta (\eta - s)y(s)\Delta s + \frac{1}{d} \int_a^T (T-s)y(s)\Delta s. \]  

Taking the nabla derivative of this expression yields \( u^{\Delta\Delta}(t) = -y(t) \), so that \( u \) given in (2.3) is a solution of (2.1) and (2.2). The rest of the proof is similar to [2, Lemma 2].

**Lemma 2.2.** If \( u(a) = 0 \) and \( u^{\Delta\Delta} \leq 0 \), then \( \frac{u(T)}{T-a} \leq \frac{u(t)}{t-a} \) for all \( t \in (a, T) \).

**Proof.** Let \( h(t) := u(t) - \frac{u(T)}{T-a} (t-a) \). Then \( h(a) = h(T) = 0 \) and \( h^{\Delta\Delta} \leq 0 \) so that \( h(t) \geq 0 \) on \( [a, T] \).

**Lemma 2.3.** Let \( 0 < \alpha < \frac{T-a}{2} \). If \( y \in C\Delta[a, T] \) and \( y \geq 0 \), the unique solution \( u \) of (2.1) and (2.2) satisfies

\[ u(t) \geq 0, \quad t \in [a, T]. \]

**Proof.** From the fact that \( u^{\Delta\Delta}(t) = -y(t) \leq 0 \), we know that the graph of \( u \) is concave down on \( (a, T) \). If \( u(T) \geq 0 \), then the concavity of \( u \) and the boundary condition \( u(a) = 0 \) imply that \( u(t) \geq 0 \) for \( t \in [a, T] \). If \( u(T) < 0 \), then we have \( u(\eta) < 0 \) and
Lemma 2.4. Let \( 0 < \alpha < \frac{T-a}{\eta-a} \). If \( y \in C_d[a,T] \) and \( y \geq 0 \), then the unique solution \( u \) as in (2.3) of (2.1) and (2.2) satisfies
\[
\inf_{t \in [\eta,T]} u(t) \geq r\|u\|,
\]
where
\[
r := \min \left\{ \frac{\alpha(T-\eta)}{d}, \frac{\alpha(\eta-a)}{T-a}, \frac{\eta-a}{T-a} \right\} > 0.
\]
where \( d := (T-a) - \alpha(\eta-a) > 0 \).

Proof. First consider the case where \( 0 < \alpha < 1 \). By the second boundary condition we know that \( u(\eta) \geq u(T) \). Pick \( t_0 \in (a,T) \) such that \( u(t_0) = \|u\| \). If \( t_0 \leq \eta < T \), then
\[
\min_{t \in [\eta,T]} u(t) = u(T)
\]
and
\[
u(t_0) \leq u(T) + \frac{u(T) - u(\eta)}{T-\eta} (a - T) = \frac{du(T)}{\alpha(T-\eta)}.
\]
Therefore
\[
\min_{t \in [\eta,T]} u(t) \geq \frac{\alpha(\eta-a)}{T-a} u(t_0),
\]
so that
\[
\min_{t \in [\eta,T]} u(t) \geq \frac{\eta-a}{T-a} \|u\|.
\]

Now consider the case \( 1 < \alpha < \frac{T-a}{\eta-a} \). The boundary condition this time implies \( u(\eta) \leq u(T) \). Set \( u(t_0) = \|u\| \). Note that by the concavity of \( u \) we have \( t_0 \in [\eta,T] \) and \( \inf_{t \in [\eta,T]} u(t) = u(\eta) \). Once again by Lemma 2.2 it follows that \( \frac{u(t_0)}{\eta-a} \geq \frac{u(T)}{T-a} \), so that
\[
\min_{t \in [\eta,T]} u(t) \geq \frac{\eta-a}{T-a} \|u\|.
\]
The proof is complete. \( \square \)

Lemma 2.5. Let \( 0 < \alpha < \frac{T-a}{\eta-a} \). If \( y \in C_d[a,T] \) and \( y \geq 0 \), then the unique solution \( u \) as in (2.3) of (2.1) and (2.2) satisfies
\[
u(t) \geq \frac{r(t-a)}{\eta-a} \|u\|, \quad t \in [a,\eta],
\]
where \( r \) is given by (2.5).

Proof. The result follows from Lemmas 2.2 and 2.4. \( \square \)

Lemma 2.6. If we define the delta and nabla polynomials \( h_2 \) and \( \hat{h}_2 \), respectively, via
\[
h_2(t,a) = \int_a^t (s-a) \Delta s, \quad \hat{h}_2(t,a) = \int_a^t (s-a) \nabla s, \quad (t,a) \in \mathbb{T} \times \mathbb{T},
\]
then the equation
\[
h_2(t,a) = \int_a^t (t-s) \nabla s
\]
holds for \( (t,a) \in \mathbb{T} \times \mathbb{T} \).
Proof. Following Bohner and Peterson [7, Section 1.6] and Anderson [1, Section 2], define the delta and nabla polynomials \( h_2 \) and \( h_3 \), respectively, as above in (2.7). Thus, by [1, Theorem 9],

\[
h_2(t, a) = \bar{h}_2(a, t) = \int_0^t (s - t) \nabla s = \int_t^0 (t - s) \nabla s.
\]

The proof is complete. □

As examples of Lemma 2.6, we have

\[
\begin{align*}
T &= \mathbb{R} \ (\text{differential equations}) : \quad h_2(t, a) = \frac{1}{2} (t - a)^2, \\
T &= h^2 \ (\text{difference equations}) : \quad h_2(t, a) = \frac{1}{2} (t - a) (t - a - h), \quad h > 0, \\
T &= q^2 \ (\text{quantum equations}) : \quad h_2(t, a) = \frac{1}{1 + q} (t - a) (t - qa), \quad q > 1, \\
T &= \mathbb{N}^2 \ (\text{squared differences}) : \quad h_2(t, a) = \frac{1}{6} \sqrt{t - \bar{a}} \frac{\sqrt{t - \bar{a}}}{\sqrt{t - 1}} \cdot \left( 3t + 3a + 6\sqrt{at} + \sqrt{a} - \sqrt{t} - 1 \right).
\end{align*}
\]

Lemma 2.7. If the function \( w \) is given by

\[
w(t) = -h_2(t, a) + \frac{t - a}{d} [h_2(T, a) - \lambda h_2(\eta, a)],
\]

where \( d = (T - a) - \lambda (\eta - a) > 0 \) and \( h_2(t, a) \) is from (2.8), then we have the following conclusions:

(i) \( w(a) = 0 \), \( z w(\eta) = w(T) \), \( w^{\Delta \nabla}_\rho(t) \equiv -1 \);

(ii) \( w(t) \leq \frac{\bar{d}}{2} [h_2(T, a) - \lambda h_2(\eta, a)] \leq \frac{\bar{d}}{2} [h_2(T, a) - \lambda h_2(\eta, a)] \) for all \( t \in [a, T]_\tau \).

Proof. From Lemmas 2.1 and 2.3 we have that \( w \geq 0 \) on \( [a, T]_\tau \) and

\[
w(t) = -\int_a^t (T - s) \nabla s - \frac{\lambda (t - a)}{d} \int_a^\eta (\eta - s) \nabla s + \frac{t - a}{d} \int_a^T (T - s) \nabla s,
\]

which is (2.9). Part (ii) follows from the fact that \( -h_2(t, a) \leq 0 \) for \( t \in [a, T]_\tau \). □

3. Existence of two positive solutions

We will employ Theorem 1.1 to establish the existence of at least two positive solutions for the second-order three-point boundary value problem (1.1) and (1.2). We will assume the following conditions.

\begin{align*}
(C_1) & \quad \text{There exists a constant } M > 0 \text{ such that } f(t, u) \geq -M \text{ for } (t, u) \in [a, T]_\tau \times [0, \infty). \\
(C_2) & \quad \text{There exist two real constants } b, c \in (0, \infty) \text{ such that } \\
& \quad 0 < f(t, u) \leq b \quad \text{for } (t, u) \in [a, T]_\tau \times [0, c]. \\
(C_3) & \quad \text{or } L := \min \{ \frac{b}{c}, c \} \text{ and } M_1 := \max \{ f(t, u) + M : (t, u) \in [a, T]_\tau \times [0, 2] \}, \text{ the parameter } \lambda \text{ satisfies } \\
& \quad 0 < \lambda \leq \tau := \frac{rd}{(T - a) [h_2(T, a) - \lambda h_2(\eta, a)]} \cdot \min \left\{ \frac{r}{M_1}, \frac{2L}{M_1}, \frac{L}{M_2} \right\},
\end{align*}

where \( r \) is given in (2.5), and \( M_2 := \max \{ f(t, u) : (t, u) \in [a, T]_\tau \times [0, L] \} \).

\begin{align*}
(C_4) & \quad \text{There exists } R > 2 \text{ such that } f(t, u) + M \geq Nu \text{ for } t \in [\eta, T]_\tau \text{ and } u \geq \frac{1}{2} R \tau^2, \text{ where } \\
& \quad N \geq \frac{2d}{\lambda \sqrt{(\eta - a) h_2(T, \eta)}} \text{ for fixed } \lambda \in [0, \tau].
\end{align*}

Remark 3.1. It follows from \( C_2 \) and the continuity of \( f \) that

\[
\lim_{u \to 0^+} \frac{f(t, u)}{u} = \infty \text{ uniformly on } [a, T]_\tau.
\]

Theorem 3.2. Suppose \( C_1 \)–\( C_4 \) hold. Then problem (1.1) and (1.2) has at least two positive solutions \( u_1 \) and \( u_2 \), where \( \| u_1 \| \geq r \) and \( \| u_2 \| \leq L < r/2 \).
Proof. Define the cone
\[ \mathcal{P} = \left\{ u : u \in C[a,T] : u(t) \geq 0, t \in [a,T] \cap \mathbb{R}^+ \right\} \]
and let \( z = \bar{M}w \), where \( r \) and \( w \) are given via (2.5) and (2.9), respectively. It is straightforward to see that (1.1) and (1.2) has a positive solution \( u \) if and only if \( u = u + z \) is a solution of
\begin{align}
\dot{u}^w(t) + \frac{\alpha(t-a)}{d} \dot{u}(t) + g(t, u(t)) &= 0, \quad t \in (a,T), \\
u(a) = 0, \quad \bar{u}(\eta) &= \bar{u}(T)
\end{align}
(3.1) (3.2)
with \( \bar{u} > z \) on \( (a,T) \), where \( g : [a,T] \times \mathbb{R}^+ \to \mathbb{R}^+ \) is defined by
\[ g(t,u) = \begin{cases} f(t,u) + M(t) & u \in [a,T] \times [0,\infty), \\
0 & u \in [a,T] \times (-\infty,0). \end{cases} \]
For \( u \in \mathcal{P} \), denote by \( Au \) the unique solution of (3.1) and (3.2), so that by Lemma 2.1 we have
\[ Au(t) = -\int_a^t (t-s)g(s,u(s)-z(s))\nabla s - \frac{\alpha(t-a)}{d} \int_a^t (\eta-s)\dot{g}(s,u(s)-z(s))\nabla s + \frac{t-a}{d} \int_a^t (T-s)\dot{g}(s,u(s)-z(s))\nabla s. \]
By Lemmas 2.3 and 2.4 we see that \( A(\mathcal{P}) \subset \mathcal{P} \); moreover, \( A \) is completely continuous by an application of the Arzelà–Ascoli theorem. Let the uniformly continuous convex functional \( \psi : \mathcal{P} \to [0,\infty) \) be defined by
\[ \psi(u) = \max_{t \in [a,T]} u(t), \quad u \in \mathcal{P}. \]
Then \( \psi(0) = 0 \) and \( \psi(u) > 0 \) for \( u \neq 0 \). Define the sets
\[ \Omega_1 = \{ u \in C[a,T] : \psi(u) < 2r \} \quad \text{and} \quad \Omega_2 = \{ u \in C[a,T] : \psi(u) < Rr \}. \]
Clearly \( \Omega_1 \) and \( \Omega_2 \) are bounded open sets in \( C[a,T] \), with \( 0 \in \Omega_1 \) and \( \Omega_1 \subset \Omega_2 \).
If \( u \in \mathcal{P} \cap \Omega_1 \), then
\[ \|u\| \leq \frac{1}{r} \min_{t \in [a,T]} u(t) \leq \frac{1}{r} \max_{t \in [a,T]} u(t) = \frac{1}{r} \psi(u) < 2. \]
which implies that \( \mathcal{P} \cap \Omega_1 \) is bounded; similarly, \( \mathcal{P} \cap \Omega_2 \) is also bounded. If \( u \in \mathcal{P} \cap \partial \Omega_1 \), then \( \psi(u) = 2r \), and thus \( \|u\| < 2 \).
Then we have
\[ \psi(Au) = \|Au\| = \max_{t \in [a,T]} \left( -\int_a^t (t-s)g(s,u(s)-z(s))\nabla s - \frac{\alpha(t-a)}{d} \int_a^t (\eta-s)\dot{g}(s,u(s)-z(s))\nabla s \right) \]
\[ + \frac{t-a}{d} \int_a^t (T-s)\dot{g}(s,u(s)-z(s))\nabla s \]
\[ \leq \max_{t \in [a,T]} \left( -\int_a^t (t-s)\nabla s - \frac{\alpha(t-a)}{d} \int_a^t (\eta-s)\nabla s + \frac{t-a}{d} \int_a^t (T-s)\nabla s \right) \]
\[ \leq \bar{M}_1 \max_{t \in [a,T]} w(t) \]
\[ \leq \bar{M}_1 \frac{t-a}{d} \left| h_2(T,a) - \frac{\alpha(t-a)}{d} \right| \leq \frac{2}{r} = \psi(u). \]
This shows that \( \psi(Au) \leq \psi(u) \) for all \( u \in \mathcal{P} \cap \partial \Omega_2 \).
If \( u \in \mathcal{P} \cap \Omega_2 \), then \( \psi(u) = rR \), so that \( rR \leq \|u\| \leq R \). Consequently, we can see that \( \inf_{u \in \mathcal{P} \cap \partial \Omega_2} \psi(u) > 0 \), and for \( u \in \mathcal{P} \cap \partial \Omega_2 \) we have
\[ z(s) = \bar{M}w(s) \leq \bar{M} \frac{t-a}{d} \left| h_2(T,a) - \frac{\alpha(t-a)}{d} \right| \leq \frac{2}{r} \leq \frac{1}{R} \|u\| \leq \frac{1}{R} u(s) \]
for \( s \in [\eta,T] \). Therefore
\[ u(s) - z(s) \geq \left( 1 - \frac{1}{R} \right) u(s), \quad s \in [\eta,T]. \]
(3.4)
Considering (3.4) and Lemma 2.4, we conclude that
\[ u(s) - z(s) \geq \frac{1}{2} u(s) \geq \frac{1}{2} r\|u\| \geq \frac{1}{2} Rr^2, \quad s \in [\eta,T]. \]
This together with \( C_d \) implies that
\[ g(s,u-z) = f(s,u-z) + M \geq N(u-z) \geq \frac{1}{2} Rr^2 N, \quad s \in [\eta,T]. \]
Moreover, we also have from (C4) that
\[
\psi(Au) = \max_{t \in [a,T]} Au(t) \geq Au(\eta)
\]
\[
= - \int_a^\eta (\eta - s)\dot{g}(s,u(s) - Z(s))\nabla s - \frac{\alpha(\eta - a)}{d} \int_a^\eta (\eta - s)\dot{g}(s,u(s) - Z(s))\nabla s + \frac{\eta - a}{d} \int_a^T (T - s)\dot{g}(s,u(s) - Z(s))\nabla s
\]
\[
= \frac{T - a}{d} \int_a^\eta (\eta - s)\dot{g}(s,u(s) - Z(s))\nabla s + \frac{\eta - a}{d} \int_a^T (T - s)\dot{g}(s,u(s) - Z(s))\nabla s
\]
\[
= \frac{T - a}{d} \int_a^\eta (s - a)\dot{g}(s,u(s) - Z(s))\nabla s + \frac{\eta - a}{d} \int_a^T (T - s)\dot{g}(s,u(s) - Z(s))\nabla s \geq \frac{\eta - a}{d} \int_a^T (T - s)\frac{1}{2}\dot{R}^2\nabla s
\]
\[
= \frac{\eta - a}{2d} \frac{\dot{R}^2}{N}\eta_{\eta T}[(T,\eta)_{\eta T}] \geq \dot{R} = \psi(u).
\]
Thus, \(\psi(Au) \geq \psi(u)\) for all \(u \in \mathcal{P} \cap \Omega_2\). It then follows from the first part of Theorem 1.1 that A has a fixed point \(u \in \mathcal{P} \cap (\Omega_2 \setminus \Omega_1)\) such that
\[
2r \leq \psi(\bar{u}) \leq rR, \quad \text{and thus } 2r \leq \|\bar{u}\| \leq R.
\]
Moreover, by combining (3.5) with (C3), and using Lemmas 2.4 and 2.7, we have that
\[
\ddot{u}(t) \geq r\|\bar{u}\| \geq 2r^2 \geq \frac{2M}{d}(T - a)[h_2(T, a) - \alpha h_2(\eta, a)] \geq 2\lambda M\bar{w}(t), \quad t \in [\eta, T].
\]
In addition, for \(t \in [a, \eta]_T\), by Lemma 2.5 and (C3) we have
\[
\ddot{u}(t) \geq \frac{r(t - a)}{\eta - a} \|\bar{u}\| \geq \frac{2r^2(t - a)}{\eta - a} \geq \frac{2M(t - a)}{d(\eta - a)}(T - a)[h_2(T, a) - \alpha h_2(\eta, a)] \geq 2\lambda M\bar{w}(t).
\]
As a consequence of (3.6) and (3.7) we see that
\[
\ddot{u}(t) \geq 2\lambda M\bar{w}(t) = 2\bar{z}(t), \quad t \in [a, T].
\]
Hence, \(u = \ddot{u} - z\) is a positive solution of (1.1) and (1.2). In addition, from (3.5) and (3.8) it follows that
\[
\|u_1\| \geq \frac{1}{2} \|\ddot{u}\| \geq r.
\]
To find the second positive solution of (1.1) and (1.2), we begin by setting
\[
f^*(t, u) = \begin{cases} f(t, u) : (t, u) \in [a, T] \times [0, 0], \\ f(t, c) : (t, u) \in [a, T] \times [c, \infty]. \end{cases}
\]
By (C2) we then have that \(0 < f^*(t, u) < b\) for \((t, u) \in [a, T] \times [0, 0]\). If we consider the auxiliary dynamic equation
\[
u^\lambda(t) + f^*(t, u(t)) = 0, \quad t \in (a, T)_T
\]
with familiar boundary conditions
\[
u(a) = 0, \quad \nu(\eta) = u(T),
\]
we know that solutions to the problem (3.11) and (3.12) are equivalent to those of the operator equation \(u = Fu\), where
\[
Fu(t) = - \int_a^t (t - s)f^*(s, u(s))\nabla s - \frac{\alpha(t - a)}{d} \int_a^t (\eta - s)f^*(s, u(s))\nabla s + \frac{t - a}{d} \int_a^T (T - s)f^*(s, u(s))\nabla s.
\]
It is straightforward that \(F : \mathcal{P} \rightarrow \mathcal{P}\) is completely continuous and \(f(\mathcal{P}) \subset \mathcal{P}\). Using Remark 3.1 we have that
\[
\lim_{u \rightarrow 0} f^*(t, u) = \infty \quad \text{uniformly on } [a, T].
\]
Thus if there exists a constant \(0 < \ell < L\) for the \(L \in (C3)\) such that \(f^*(t, u) \geq 2\beta\) for \((t, u) \in [a, T] \times [0, \ell]\), where \(\beta > 0\) is chosen so that
\[
\beta \frac{\alpha}{d} \geq \frac{\lambda h_2(T, \eta)}{\eta T} \geq 1.
\]
Choose
\[
\Omega_3 = \{u \in C[a, T] : \psi(u) < \ell r\}, \quad \Omega_4 = \{u \in C[a, T] : \psi(u) < \ell r\}
\]
Then \(\mathcal{P} \cap \Omega_3\) and \(\mathcal{P} \cap \Omega_4\) are bounded open sets in \(C[a, T]\) with \(\inf_{u \in \mathcal{P} \cap \Omega_3, \Omega_4} \psi(u) > 0\).
If \( u \in \mathcal{P} \cap \partial \Omega_3 \), then from Lemmas 2.3 and 2.7 we have that
\[
\psi(Fu) = \max_{t \in [0,T]} Fu(t) = \max_{t \in [0,T]} \left( \int_{a}^{T} (s - T) f(s, u(s)) \, ds - \frac{\alpha(t - a)}{d} \int_{a}^{T} \int_{a}^{\eta} \rho(s, \eta, u(s)) \, d\eta \, ds + \frac{t - a}{d} \int_{a}^{T} \rho(T, \eta, u(s)) \, d\eta \right)
\]
\[
\leq \max_{t \in [0,T]} \left( \int_{a}^{T} (s - T) f(s, u(s)) \, ds - \frac{\alpha(t - a)}{d} \int_{a}^{T} \rho(s, \eta, u(s)) \, d\eta \, ds + \frac{t - a}{d} \int_{a}^{T} \rho(T, \eta, u(s)) \, d\eta \right)
\]
\[
\leq \lambda M_2 \max_{t \in [0,T]} w(t)
\]

This shows that \( \psi(Fu) \leq \psi(u) \) for all \( u \in \mathcal{P} \cap \partial \Omega_3 \).

If \( u \in \mathcal{P} \cap \partial \Omega_2 \), then \( r \leq ||u|| \leq \epsilon \), and we have
\[
\psi(Fu) = \max_{t \in [0,T]} Fu(t) \geq Fu(T)
\]
\[
= - \int_{a}^{T} (s - T) f(s, u(s)) \, ds + \frac{\alpha(t - a)}{d} \int_{a}^{T} \int_{a}^{\eta} \rho(s, \eta, u(s)) \, d\eta \, ds + \frac{t - a}{d} \int_{a}^{T} \rho(T, \eta, u(s)) \, d\eta
\]
\[
\geq \lambda \beta \rho u(\eta - a) ||u|| \int_{a}^{T} \int_{a}^{\eta} \rho(s, \eta, u(s)) \, d\eta \, ds + \frac{\alpha(T - a)}{d} \int_{a}^{T} f(s, u(s)) \, ds \geq \frac{\alpha(T - a)}{d} \int_{a}^{T} f(s, u(s)) \, ds \geq \frac{\alpha(T - a)}{d} \int_{a}^{T} f(s, u(s)) \, ds
\]

Therefore, \( \psi(Fu) \geq \psi(u) \) for all \( u \in \mathcal{P} \cap \partial \Omega_2 \). It then follows from the second part of Theorem 1.1 that the problem given in (3.11) and (3.12) has a positive solution \( u_2 \) satisfying \( r \leq ||u_2|| \leq L \). Consequently, \( ||u_2|| \leq L < r/2 \). In light of (C3) and (3.10) we conclude that \( u_2 \) is also a solution of the original problem 1.1 and 1.2. From (C3) and (3.9) we then have that the problem (1.1) and (1.2) has two distinct positive solutions \( u_1 \) and \( u_2 \).}

**Corollary 3.3.** Suppose (C1) and (C4) hold. If
\[
0 < \lambda \leq \frac{r d}{(T - a) h_2(T, a) - \delta h_2(\eta, a)} \min \left\{ r, \frac{2}{M} \right\}
\]
then problem (1.1) and (1.2) has at least one positive solution \( u_1 \) with \( ||u_1|| \geq r \).

**Remark 3.4.** The new methods employed on arbitrary time scales in this paper extend and generalize those found in [15]. In a similar way we might discuss the existence of positive solutions for a semi-positone problem with altered condition at the left end point, such as
\[
u^T(t) + \int_{a}^{T} f(t, u(t)) \, dt = 0, \quad u^h(a) = 0, \quad \xi u(\eta) = u(T).
\]

With appropriate conditions on \( \lambda \), results similar to those given in Theorem 3.2 and Corollary 3.3 can be derived.

### 4. Example applying Theorem 3.2

For a nice example for \( T = \mathbb{R} \) set on the unit interval, see [15, Example 3.1]. Let us apply Theorem 3.2 to the following three-point boundary value problem on time scales given by
\[
u^T(t) + \int_{a}^{T} f(t, u(t)) \, dt = 0, \quad t \in (a, T),
\]
\[
u(a) = 0, \quad \xi u(\eta) = u(T),
\]
where \( a = 1, \quad \eta = 4, \quad T = 128, \quad x = 2, \quad d = 121, \quad T \) is any time scale with \( a, \eta, T \in T \), and
\[
h(u) = \begin{cases} \frac{51}{2} (u - \frac{25}{3})^2 - \frac{287}{3}, & u \in [0, 1], \\ u^2, & u > 1. \end{cases}
\]

**Example 4.1.** If \( \lambda \in (0, \tau) \), then problem (4.1) and (4.2) has at least two positive solutions \( u_1 \) and \( u_2 \) with \( ||u_1|| \geq \frac{5}{12} \) and \( ||u_2|| \leq \frac{5}{12} \), where
\[
\tau = \frac{55539}{587885921 e_{128, 128} [h_{128, 128} - 2 h_{128, 128}]}.
\]
Proof. Setting \( f(t, u) = e_1(t, 1)h(u) \) and \( M = \frac{2\pi}{12} e_1(128, 1) \), we see that \( r = 127 \), \( M_1 = (4 + \frac{2\pi}{12}) e_1(128, 1) \), \( b = \frac{1}{2} e_1(128, 1) = M_2 \). \( c = \frac{1}{127} \), \( L = \frac{1}{254} \), and \( \tau \) is given by (4.3). Consequently we have that

\[
\begin{align*}
\frac{f(t, u)}{M} & \leq 0 \quad \text{for } \{1, 128\} \times [0, \infty), \\
0 & < \frac{f(t, u)}{b} \quad \text{for } \{1, 128\} \times [0, c].
\end{align*}
\]

Since \( \lim_{t \to \infty} \frac{e_1(t, 1)h(u)}{M} = 0 \) uniformly on \([1, 128]\), there exists \( R > 2 \) such that \( f(t, u) + M \geq Nu \) for all \( t \in [4, 128] \) and \( u \geq \frac{1}{2} R^2 = \frac{9}{254} R \), where

\[
N = \frac{30734}{9h_2(128, 1)}
\]

for fixed \( \varepsilon \in [0, \tau] \).

As all of the conditions of Theorem 3.2 are satisfied, problem (4.1), (4.2) has at least two positive solutions \( u_1 \) and \( u_2 \) with \( \|u_1\| \geq \frac{1}{254} \) and \( \|u_2\| \leq \frac{1}{254} \). Note that 1, 128 \( \in \mathbb{T} \) for the following three key time scales:

\[
\begin{align*}
T = \mathbb{R} \quad \text{(differential equations)}: h_2(t, 1) = \frac{1}{2} (t - 1)^2, \\
T = \mathbb{Z} \quad \text{(difference equations)}: h_2(t, 1) = \frac{1}{2} (t - 1)(t - 2), \\
T = 2^t \quad \text{(quantum equations)}: h_2(t, 1) = \frac{1}{3} (t - 1)(t - 2),
\end{align*}
\]

with the convention that \( \prod_{k=0}^{-1}(1 + 2^k) = 1 \); this exponential for \( q^t \) is also known as a \( q \)-Pochhammer function. \( \square \)

References