Solvability for a third-order three-point BVP on time scales

Douglas R. Anderson a,*, George Smyrlis b

a Department of Mathematics and Computer Science, Concordia College, Moorhead, MN 56562, USA
b Department of Mathematics, Technological Educational Institute of Athens, Ag. Spyridonos Street, Egaleo 12210, Athens, Greece

ARTICLE INFO

Article history:
Received 13 July 2008
Accepted 12 November 2008

Keywords:
Leray–Schauder nonlinear alternative
Timescales
Third order
Three point
Green function
Existence

ABSTRACT

We are concerned with the existence of a nontrivial solution to a nonlinear third-order three-point boundary-value problem on general time scales. Using the corresponding Green function, without a non-negative or monotone-type assumption on the nonlinearity, we obtain sufficient conditions for the existence of at least one nontrivial solution, using the Leray–Schauder nonlinear alternative theorem. This paper extends recent results in differential equations to difference equations, quantum equations, and general dynamic equations on time scales.

© 2009 Elsevier Ltd. All rights reserved.

1. Introduction

We are concerned with the existence of at least one nontrivial solution to the nonlinear third-order three-point boundary-value problem on general time scales given by

\[(px^{\Delta^2})^{\nabla} + f(t, x(t), x^{\Delta^3}(t)) = 0, \quad t \in [t_1, t_3]_T,\]

\[x(\rho(t_1)) = x^{\Delta^3}(\rho(t_1)), \quad x^{\Delta^3}(\sigma(t_3)) = \alpha x^{\Delta^3}(t_2),\]

where: \(p\) is a right-dense continuous, real-valued function with \(0 < p(t) \leq 1\) on \(T\); \(f : T \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous; the boundary points from \(T\) satisfy \(t_1 < t_2 < t_3\) such that the constants \(d\) and \(\alpha\) satisfy

\[d := \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)} - \alpha \int_{\rho(t_1)}^{t_2} \frac{\Delta \tau}{p(\tau)} > 0 \quad \text{and} \quad 1 < \alpha < \frac{\int_{\rho(t_1)}^{t_2} \frac{\Delta \tau}{p(\tau)}}{\int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)}}.\]

If \(T = \mathbb{R}\), then (1.1) and (1.2) is the ordinary third-order three-point boundary value problem

\[(px^{\prime\prime})^\prime(t) + f(t, x(t), x^{\prime}(t)) = 0, \quad t \in [t_1, t_3]_\mathbb{R},\]

\[x(t_1) = x^{\prime}(t_1), \quad x(t_3) = \alpha x^{\prime}(t_2).\]

If \(T = \mathbb{Z}\), then (1.1) and (1.2) is the discrete third-order three-point boundary value problem

\[\nabla (px^{\Delta^2})^\nabla(t) + f(t, x(t), x^{\Delta^3}(t)) = 0, \quad t \in \{t_1, t_1 + 1, t_1 + 2, \ldots, t_3\},\]

\[x(t_1 - 1) = 0 = \Delta x(t_1 - 1), \quad \Delta x(t_3 + 1) = \alpha \Delta x(t_2).\]

* Corresponding author.
E-mail addresses: andersod@cord.edu (D.R. Anderson), gsmyrilis@teiath.gr (G. Smyrlis).
where $\Delta y(t) = y(t + 1) - y(t)$ and $\nabla y(t) = y(t) - y(t - 1)$. As a final illustration, if $T$ is a quantum time scale for some real $q > 1$, then (1.1) and (1.2) is the third-order three-point quantum boundary value problem
\[
D^q (pD_q (D_q x))(t) + f \left( t, x(t), D_q x(t) \right) = 0, \quad t \in \{ t_1, t_3 \}_T, \\
x(t_1/q) = 0 = D_q x(t_1/q), \quad D_q x(t_3) = \alpha D_q x(t_2),
\]
where the quantum derivatives are given by the difference quotients
\[
D_q x(t) = \frac{y(qt) - y(t)}{(q - 1)t} \quad \text{and} \quad D^q y(t) = \frac{y(t) - y(t/q)}{(1 - 1/q)t},
\]
respectively, and $T = \{ 0, \ldots, q^{-2}, q^{-1}, 1, q, q^2, \ldots \}$.

Third-order differential equations, though less common in applications than even-order problems, nevertheless do appear, for example, in the study of quantum fluids and gravity driven flows. Here we approach a third-order three-point problem on general time scales, namely on any nonempty closed subset of the real line, to include the discrete, continuous, and quantum calculus as special cases. Of late there have been several papers on third-order boundary value problems. Hopkins and Kosmatov [1]; Li [2]; Liu, Ume, and Kang [3,4]; and Minghe and Chang [5] have all recently considered third-order problems. All of these papers, however, were two-point problems with $T = \mathbb{R}$. Graef and Yang [6], Sun [7], and Wong [8] consider three-point focal problems, while Palamides and Smyrlis [18] consider the three-point boundary conditions

\[
x(0) = x'(n) = x(1) = 0, \quad T = [0, 1]_R.
\]

On general time scales there are also a few papers on third-order problems. Sun [9] considers a third-order two-point boundary value problem; a couple of papers on third-order three-point boundary value problems considered on general time scales are [10,11] in the right-focal case, and [12]. Note that boundary value problems on time scales that utilize both delta and nabla derivatives, such as the one here, were first introduced by Atici and Guseinov [13]. For more on existence of solutions to boundary value problems, see [14, Chapters 4 and 6-9], the text by Deimling [15], and Zhang and Liu [16].

Problem (1.1) and (1.2) is an extension of the unit interval boundary value problem [17]
\[
\begin{align*}
\chi'''(t) + f \left( t, x(t), x'(t) \right) &= 0, \quad t \in (0, 1)_R, \\
\chi(0) &= 0 = \chi'(1), \quad \alpha \chi'(t_2),
\end{align*}
\]
to arbitrary time scales; in other words, take $T = \mathbb{R}$, $p \equiv 1$, $t_1 = 0$, $t_2 = t$ and $\alpha = 1$ in (1.1) and (1.2) to get the results in [17]. One could also consider a third-order problem with derivatives in the order of nabla, delta, beta, but the results would be similar; other permutations of nablas and/or deltas lead to a Green function that is less easy to calculate.

2. Preliminary results

Underlying our technique will be the Green function for the homogeneous third-order three-point boundary-value problem
\[
-(p x^\Delta \nabla) y(t) = 0, \quad t \in [t_1, t_3]_T, \\
x(p(t_1)) = 0 = x'(p(t_1)), \quad x'(p(t_3)) = \alpha x'(t_2).
\]
\[
\text{The Green function for (2.1) and (2.2) is well defined, nonnegative, and bounded above on } [\rho(t_1), \sigma^2(t_3)]_T \times [t_1, \sigma(t_3)]_T, \text{ as related in the following lemmas.}
\]

Lemma 2.1 (Lemma 2.1 [12]). For $y \in C_{[\rho(t_1), \sigma(t_3)]_T}$, the boundary value problem
\[
(p x^\Delta \nabla) y(t) + y(t) = 0, \quad t \in [t_1, t_3]_T, \\
x(p(t_1)) = 0 = x'(p(t_1)), \quad x'(p(t_3)) = \alpha x'(t_2)
\]
has a unique solution $x(t) = \int_{p(t_1)}^{\sigma(t_3)} G(t, s)y(s)\nabla s$, where the Green function corresponding to the problem (2.1) and (2.2) is given by
\[
G(t, s) = \begin{cases}
\frac{1}{d} \left( \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \xi \frac{\rho(t)}{p(\tau)} \Delta \xi - \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \right) : & s \leq \min\{t_2, t\} \\
\frac{1}{d} \left( \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \xi \frac{\rho(t)}{p(\tau)} \Delta \xi \right) - \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \Delta \xi : & t_2 \leq s \leq t \\
\frac{1}{d} \left( \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \xi \frac{\rho(t)}{p(\tau)} \right) : & t \leq s \leq t \\
\frac{1}{d} \left( \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \tau \frac{\rho(t)}{p(\tau)} \int_{\rho(t_1)}^{\sigma(t_3)} \Delta \xi \frac{\rho(t)}{p(\tau)} \Delta \xi \right) : & \max\{t_2, t\} \leq s
\end{cases}
\]
for all $(t, s) \in [\rho(t_1), \sigma^2(t_3)]_T \times [t_1, \sigma(t_3)]_T$. 

Remark 2.2. If $T = \mathbb{R}$, $p \equiv 1$, $t_1 = 0$, $t_2 = \eta$, and $t_3 = 1$, then $d = 1 - \alpha \eta$ and the Green function corresponding to problem (2.1) and (2.2) is given by

$$G(t, s) = \frac{1}{2(1 - \alpha \eta)} \begin{cases} (2ts - s^2)(1 - \alpha \eta) + t^2s(\alpha - 1) : & s \leq \min\{\eta, t\} \\ t^2(1 - \alpha \eta) + t^2s(\alpha - 1) : & t \leq s \leq \eta \\ (2ts - s^2)(1 - \alpha \eta) + t^2(\alpha \eta - s) : & \eta \leq s \leq t \\ t^2(1 - s) : & \max\{\eta, t\} \leq s \end{cases}$$

for all $(t, s) \in [0, 1) \times [0, 1]$; see [17].

If $T = h\mathbb{Z}$ for any $h > 0$ and $p \equiv 1$, then assumption (1.3) becomes

$$d = t_3 - t_1 + 2h - \alpha (t_2 - t_1 + h) > 0 \quad \text{and} \quad 1 < \alpha < \frac{t_3 - t_1 + 2h}{t_2 - t_1 + h},$$

and the Green function corresponding to problem (2.1) and (2.2) is given by

$$G(t, s) = \frac{1}{2d} \begin{cases} (t + h - t_1)(t - t_1) \{[t_3 + h - s - \alpha(t_2 - s)] - d(t - s)(t - s - h) : & s \leq \min\{t_2, t\} \\ (t + h - t_1)(t - t_1) \{[t_3 + h - s - \alpha(t_2 - s)] : & t \leq s \leq t_2 \\ (t + h - t_1)(t - t_1)(t_2 + h - s) - d(t - s)(t - s - h) : & t_2 \leq s \leq t \\ (t + h - t_1)(t - t_1)(t_3 + h - s) : & \max\{t_2, t\} \leq s \end{cases}$$

for all $(t, s) \in [t_1 - h, t_3 + 2h]_{h\mathbb{Z}} \times [t_1, t_3 + h]_{h\mathbb{Z}}$.

Lemma 2.3 (Lemma 2.2[12]). Assume (1.3). The Green function (2.5) corresponding to the homogeneous boundary value problem (2.1) and (2.2) satisfies

$$0 \leq G(t, s) \leq \frac{(\alpha + 1) (\sigma^2(t_2) - \rho(t_1))}{d} g(s, s), \quad (t, s) \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \times [t_1, \sigma(t_3)]_{\mathbb{T}},$$

where $g$ is given by

$$g(t, s) := \left(\int_{\rho(t_1)}^{\min\{t, s\}} \frac{\Delta \tau}{p(t)} \right) \left(\int_{\max\{t, s\}}^{\sigma(t_3)} \frac{\Delta \tau}{p(t)} \right)$$

for all $t, s \in [t_1, \sigma(t_3)]_{\mathbb{T}}$.

3. Main existence results

To establish an existence result we will employ the following fixed point theorem, which can be found in Deimling [15].

Theorem 3.1 ([15]). Let $X$ be a real Banach space, $\Omega$ an open bounded subset of $X$ with $0 \in \Omega$, and let $L : \overline{\Omega} \rightarrow X$ be a completely continuous operator. Then either there exists an $x \in \partial \Omega$ and $\lambda > 1$ such that $L(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.

Let $Y$ denote the real Banach space $C[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$ with the supremum norm

$$\|x\| = \sup_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} |x(t)|.$$

In $X = C^1[\rho(t_1), \sigma(t_3)]_{\mathbb{T}}$, introduce the norm

$$\|x\|_1 := \|x\| + \|x'\| = \sup_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} |x(t)| + \sup_{t \in [\rho(t_1), \sigma(t_3)]_{\mathbb{T}}} |x'(t)|;$$

then $(X, \| \cdot \|_1)$ is a real Banach space as well.

For $x \in X$, define

$$Lx(t) := \int_{\rho(t_1)}^{\sigma(t_3)} G(t, s) f(s, x(s), x'(s)) \, ds, \quad t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}.$$

By Lemma 2.1, the boundary value problem (1.1) and (1.2) has a solution $x = x(t)$ if and only if $x$ solves the operator equation

$$x(t) = Lx(t), \quad t \in X.$$

Clearly $L(X) \subseteq X$, and $L$ is completely continuous by a standard application of the Ascoli–Arzelà theorem; thus, we seek a fixed point of $L$ in $X$. 
Theorem 3.2. Assume (1.3), \( f(s,0,0) \not\equiv 0 \), and there exist non-negative integrable functions \( u, v, w \) on \( [\rho(t_1), \sigma(t_1)] \) such that
\[
\begin{align*}
\frac{1}{d} \int_{\rho(t_1)}^{\sigma(t_1)} \left[ (\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right) g(s, s) + \alpha g(t_2, s) \right] u(s) + v(s) \, ds < 1.
\end{align*}
\]
Then (1.1) and (1.2) has at least one nontrivial solution \( x^* \) in \( X = C^1[\rho(t_1), \sigma(t_1)] \).

Proof. Set
\[
A = \frac{1}{d} \int_{\rho(t_1)}^{\sigma(t_1)} \left[ (\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right) g(s, s) + \alpha g(t_2, s) \right] u(s) + v(s) \, ds,
\]
\[
B = \frac{1}{d} \int_{\rho(t_1)}^{\sigma(t_1)} \left[ (\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right) g(s, s) + \alpha g(t_2, s) \right] w(s) \, ds,
\]
where \( g(\cdot, \cdot) \) is given in (2.6). By assumption, \( A < 1 \). Since \( f \) is continuous and \( f(s,0,0) \not\equiv 0 \), there exists \( [a, b] \subseteq [\rho(t_1), \sigma(t_3)] \) such that
\[
\min_{t \in [a,b]} |f(s,0,0)| > 0.
\]
Since \( |f(s,0,0)| \leq w(s) \) for \( s \in [\rho(t_1), \sigma(t_3)] \), we have \( B > 0 \). Set \( m := B(1 - A)^{-1} \) and \( \Omega_m := \{ x \in C^1[\rho(t_1), \sigma(t_3)] : \| x \|_1 < m \} \). If there exist \( x \in \partial \Omega_m \) and \( \lambda > 1 \) such that \( \lambda x = \lambda x \), then
\[
\lambda m = \lambda \| x \|_1 = \| \mathcal{L} x \| + \| (\mathcal{L} x)^\star \|.
\]
By definition,
\[
\| \mathcal{L} x \| = \sup_{t \in [\rho(t_1), \sigma(t_3)]} |\mathcal{L} x(t)|
\]
\[
\leq \sup_{t \in [\rho(t_1), \sigma(t_3)]} \int_{\rho(t_1)}^{\sigma(t_1)} G(t, s) \left| f(s, x(s), x^3(s)) \right| \, ds
\]

\[
\leq \frac{d}{(\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right)} \int_{\rho(t_1)}^{\sigma(t_1)} g(s, s) \left| f(s, x(s), x^3(s)) \right| \, ds
\]
\[
\leq \frac{d}{(\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right)} \int_{\rho(t_1)}^{\sigma(t_1)} g(s, s) \left| u(s) x(s) + v(s) x^3(s) + w(s) \right| \, ds
\]
\[
\leq \frac{d}{(\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right)} \| x \|_1 \int_{\rho(t_1)}^{\sigma(t_1)} g(s, s) \left| u(s) + v(s) \right| \, ds
\]
\[
+ \frac{d}{(\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right)} \int_{\rho(t_1)}^{\sigma(t_1)} g(s, s) w(s) \, ds.
\]
For \( \| (\mathcal{L} x)^\star \| \), we consider \( |(\mathcal{L} x)^\star(t)| \) for \( t \in [\rho(t_1), t_2] \) and \( t \in [t_2, \sigma(t_3)] \). If \( t \in [\rho(t_1), t_2] \), then using the appropriate branches of (2.5) we see that
\[
|(\mathcal{L} x)^\star(t)| = \left| \int_{\rho(t_1)}^{t} \left[ \frac{1}{d} \int_{s}^{t} \frac{\Delta r}{p(r)} \right] A \left[ \int_{s}^{t} \frac{\Delta r}{p(r)} \right] f(s, x(s), x^3(s)) \, ds \right|
\]
\[
\leq \frac{d}{(\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right)} \| x \|_1 \int_{\rho(t_1)}^{t} \frac{\Delta r}{p(r)} \, ds
\]
\[
+ \frac{d}{(\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right)} \int_{t_1}^{t} \frac{\Delta r}{p(r)} \, ds
\]
\[
\leq \frac{d}{(\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right)} \| x \|_1 \int_{\rho(t_1)}^{t} \frac{\Delta r}{p(r)} \, ds
\]
\[
+ \frac{d}{(\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right)} \int_{t_1}^{t} \frac{\Delta r}{p(r)} \, ds.
\]
Assume Then and one of the following conditions holds:

\[ \exists \lambda \in [\ell, < 1 ) \text{ such that } \| \lambda \| \leq 3\| \alpha \| \| s \| + \| u \| + \| v \| \| s \| . \]

It follows that

\[ \lambda = A + B/m = A + B/\lambda \]

by the choice of m, a contradiction of \( \lambda > 1 \). By Theorem 3.1, \( \mathcal{X} \) has a fixed point \( x^* \in \mathcal{X} \). Since \( f(s, 0, 0) \not\equiv 0 \), this fixed point \( x^* \in C^1[\rho(t_1), \sigma(t_3)] \) is a nontrivial solution of the third-order three-point boundary value problem (1.1) and (1.2), with \( 0 < \| x \| < \lambda \). \( \square \)

**Corollary 3.3.** Assume (1.3), \( f(s, 0, 0) \not\equiv 0 \), and there exist non-negative integrable functions \( u, v, w \) on \([\rho(t_1), \sigma(t_3)]\) such that

\[ \| f(s, x, y) \| \leq u(s)\| x \| + v(s)\| y \| + w(s) \]

and one of the following conditions holds:

1. there exist constants \( \ell, k > 1 \) with \( 1/\ell + 1/k = 1 \) such that

\[ \int_{\rho(t_1)}^{\sigma(t_3)} (u(s) + v(s))^k \| s \| < \left\{ \frac{(\sigma^2(t_3) - \rho(t_1))}{\ell} \left[ \int_{\rho(t_1)}^{\sigma(t_3)} g(s, s)^k \| s \| \right]^{1/k} + \frac{\alpha}{d} \left[ \int_{\rho(t_1)}^{\sigma(t_3)} g(t_2, s)^k \| s \| \right]^{1/k} \right\}^{-\ell}, \]

where \( g \) is given in (2.6);

2. for all \( s \in [\rho(t_1), \sigma(t_3)] \), the function \( u + v \) satisfies

\[ u(s) + v(s) < d \left( \int_{\rho(t_1)}^{\sigma(t_3)} [(\alpha + 1) (\sigma^2(t_3) - \rho(t_1)) g(s, s) + \alpha g(t_2, s)] \| s \| \right)^{-1}. \]

Then (1.1) and (1.2) has at least one nontrivial solution \( x^* \) in \( \mathcal{X} = C^4[\rho(t_1), \sigma(t_3)] \).
Proof. Let $A$ be as in the proof of Theorem 3.2; it suffices to show that $A < 1$.

(1) Using Hölder’s inequality [14, Theorem 6.13],

$$A = \frac{1}{d} \int_{\rho(t_1)}^{\sigma(t_1)} \left[ (\alpha + 1) \left( \sigma^2(t_2) - \rho(t_1) \right) g(s, s + \alpha g(t_2, s)) \right] \left[ u(s) + v(s) \right] \Delta s \leq \left[ \int_{\rho(t_1)}^{\sigma(t_1)} \left( u(s) + v(s) \right) \right]^{\frac{1}{2}} \left[ \int_{\rho(t_1)}^{\sigma(t_1)} \left( \alpha + 1 \right) \left( \sigma^2(t_2) - \rho(t_1) \right) g(s, s + \alpha g(t_2, s)) \right]^{\frac{1}{2}} \Delta s \leq 1.$$

(2) In this case it immediately follows that

$$A = \frac{1}{d} \int_{\rho(t_1)}^{\sigma(t_2)} \left[ (\alpha + 1) \left( \sigma^2(t_2) - \rho(t_1) \right) g(s, s + \alpha g(t_2, s)) \right] \left[ u(s) + v(s) \right] \Delta s < 1.$$

By Theorem 3.2, the (3, 3) problem (1.1) and (1.2) has at least one nontrivial solution $x^*$ in $X = C^0[\rho(t_1), \sigma(t_3)]$.

Theorem 3.4. Assume (1.3), $f(s, 0, 0) \neq 0$, and there exist non-negative integrable functions $u, v$ on $[\rho(t_1), \sigma(t_3)]_T$ such that

$$\begin{align*}
\text{for all } (s, x_1, x_2) \in & \left[ t_1, \sigma(t_3) \right]_T \times \mathbb{R} \times \mathbb{R}, \quad i \in \{1, 2\}, \\
\frac{1}{d} & \int_{\rho(t_1)}^{\sigma(t_3)} \left[ (\alpha + 1) \left( \sigma^2(t_2) - \rho(t_1) \right) g(s, s + \alpha g(t_2, s)) \right] \left[ u(s) + v(s) \right] \Delta s < 1.
\end{align*}$$

Then (1.1) and (1.2) has a unique nontrivial solution $x^*$ in $X = C^0[\rho(t_1), \sigma(t_3)]_T$.

Proof. We will show that in this case $L$ given in (3.1) is a contraction. To this end, let $x_1, x_2 \in X$ be fixed. We set

$$F(x_1, x_2)(s) = f \left( s, x_1(s), x_2(s) \right), \quad s \in [\rho(t_1), \sigma(t_3)]_T.$$

It follows from the hypotheses that

$$|F(x_1, x_2)(s)| \leq \|x_1 - x_2\|_1 [u(s) + v(s)], \quad s \in [\rho(t_1), \sigma(t_3)]_T.$$

Now we have

$$\|Lx_1 - Lx_2\| = \sup_{t \in [\rho(t_1), \sigma(t_3)]_T} \left| \int_{\rho(t_1)}^{\sigma(t_3)} G(t, s)F(x_1, x_2)(s) \Delta s \right| \leq \frac{(\alpha + 1) \left( \sigma^2(t_2) - \rho(t_1) \right)}{d} \int_{\rho(t_1)}^{\sigma(t_3)} g(s, s) [u(s) + v(s)] \Delta s.$$

Next, we will estimate $|\langle Lx_1 \rangle^A(t) - \langle Lx_2 \rangle^A(t)|$ for $t \in [\rho(t_1), t_2]_T$ and $t \in [t_2, \sigma(t_3)]_T$.

If $t \in [\rho(t_1), t_2]_T$, then using the appropriate branches of (2.5) we get that

$$\begin{align*}
\left| \langle Lx_1 \rangle^A(t) - \langle Lx_2 \rangle^A(t) \right| &= \left| \int_{\rho(t_1)}^{t_2} \frac{1}{d} \left( \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)} - \alpha \int_{t}^{t_2} \frac{\Delta \tau}{p(\tau)} \right) \left( \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)} \right) F(x_1, x_2)(s) \Delta s \right| \\
&\quad + \left( \int_{t}^{t_2} \left( \frac{1}{d} \left( \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)} - \alpha \int_{t}^{t_2} \frac{\Delta \tau}{p(\tau)} \right) \left( \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)} \right) \right) F(x_1, x_2)(s) \Delta s \right| \\
&\quad + \left( \int_{t_2}^{\sigma(t_3)} \left( \frac{1}{d} \left( \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)} - \alpha \int_{t}^{t_2} \frac{\Delta \tau}{p(\tau)} \right) \left( \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)} \right) \right) F(x_1, x_2)(s) \Delta s \right| \\
&\quad \leq \frac{1}{d} \int_{\rho(t_1)}^{t_2} \left( \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)} - \alpha \int_{t}^{t_2} \frac{\Delta \tau}{p(\tau)} \right) \left( \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta \tau}{p(\tau)} \right) |F(x_1, x_2)(s)| \Delta s.
\end{align*}$$
Proof.\n
We estimate the right-hand side of (1.2)

\[
\frac{1}{d} \int_{\rho(t_3)}^{\rho(t_2)} \left[ \int_{\rho(t_2)}^{t_2} \frac{\Delta \tau}{p(\tau)} \int_{t_2}^{t_2} \frac{\Delta \tau}{p(\tau)} \right] |F(x_1, x_2)(s)| \nabla s.
\]

Consequently, (2.6)

\[
\frac{1}{d} \int_{\rho(t_1)}^{\rho(t_3)} g(t_2, s) |F(x_1, x_2)(s)| \nabla s.
\]

Similarly, for \( t \in [t_2, \sigma(t_3)] \), we get that

\[
\left| (Lx)^{\alpha} - (Lx_2)^{\alpha} \right| \leq \frac{\alpha \| x_1 - x_2 \|_1}{d} \int_{\rho(t_1)}^{\rho(t_3)} g(t_2, s)[u(s) + v(s)] \nabla s.
\]

and since \( \alpha > 1 \),

\[
\left\| (Lx)^{\alpha} - (Lx_2)^{\alpha} \right\|_1 \leq \| x_1 - x_2 \|_1 \frac{1}{d} \int_{\rho(t_1)}^{\rho(t_3)} \left[ (\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right) g(s, s) + \alpha g(t_2, s) \right] [u(s) + v(s)] \nabla s.
\]

which implies that \( L \) is a contraction. Thus, by the Banach fixed point theorem \( L \) possesses a unique fixed point \( x^* \in X \).

Since \( f(s, 0, 0) \neq 0 \), this fixed point \( x^* \in C^1(\rho(t_1), \sigma(t_3)) \) is the unique nontrivial solution of the third-order three-point boundary value problem (1.1) and (1.2). \( \square \)

4. Example

In this section we consider the following example to illustrate our results.

Example 4.1. Let \( \epsilon = \epsilon \) and \( s \) for some \( 0 < \epsilon < 1 \) (say \( \epsilon = 1/N \) for some large \( N \in \mathbb{Z} \)), \( p \equiv 1 \), \( t_1 = \epsilon, t_2 = 10\epsilon \), and \( t_3 = 1 - \epsilon \).

If \( 1 < \alpha < \frac{1}{N^2 + 15N - 100} \) for \( N \geq 54 \), then the boundary value problem

\[
\begin{align*}
\nabla^2 x(t) + t \sin t x(t) - t \cos(\Delta x(t)) + t^2 = 0, \quad t \in [\epsilon, 1 - \epsilon], \\
x(0) = 0 = \Delta x(0), \quad \Delta x(1) = \alpha \Delta x(10\epsilon),
\end{align*}
\]

(4.1) \( x(0) = 0 = \Delta x(0), \) \( \Delta x(1) = \alpha \Delta x(10\epsilon), \) (4.2) and \( \Delta x(t) = \frac{\Delta x(t)}{t}, \) it \( \frac{\Delta x(t)}{t}. \) has a nontrivial solution.

Proof. By Remark 2.2, choose \( \alpha \) such that

\[
d = 1 - 10\epsilon \alpha > 0 \quad \text{and} \quad 1 < \alpha < \frac{1}{10\epsilon}.
\]

Then the Green function corresponding to problem (2.1), (2.2) is given by

\[
G(t, s) = \frac{1}{2d} \begin{cases} 
\left( \frac{t}{10\epsilon - s} \right) - \left( \frac{t}{s} \right), & s \leq \min(10\epsilon, t) \\
\left( \frac{t}{10\epsilon - s} \right) - \left( \frac{t}{s} \right), & t \leq s \leq 10\epsilon \\
\left( t - (10\epsilon - s) \right), & 10\epsilon \leq s \leq t \\
\left( \frac{t}{10\epsilon - s} \right) - \left( \frac{t}{s} \right), & s \leq \min(10\epsilon, t)
\end{cases}
\]

for all \((t, s) \in [0, 1 + \epsilon] \times [\epsilon, 1]\). Since

\[
f(s, x, y) = s \sin s x - s \cos y + s^2,
\]

\( f(0, 0) = \epsilon^2 - 0 \neq 0 \) and \( |f(s, x, y)| \leq s|x| + (s + 1)|y| + 1 + s^2 \) for all \((s, x, y) \in [\epsilon, 1] \times \mathbb{R} \times \mathbb{R} \). Moreover, we have

\[
A = \frac{1}{d} \int_{\rho(t_1)}^{\rho(t_3)} \left[ (\alpha + 1) \left( \sigma^2(t_3) - \rho(t_1) \right) g(s, s) + \alpha g(t_2, s) \right] [u(s) + v(s)] \nabla s
\]

\[
= \frac{\epsilon}{d} \sum_{k=1}^{N} (2k\epsilon + 1) \left( \alpha + 1 \right) (\epsilon + 1) g(k\epsilon, k\epsilon) + \alpha g(10\epsilon, k\epsilon)
\]
\[ \sum_{k=1}^{N} \frac{2k \varepsilon + 1}{k!} \left( \alpha + 1 \right) (1 - k \varepsilon) + \alpha (1 - 10 \varepsilon) \]

\[ \frac{N^{3}(1 + \alpha) + N^{2}(1 + 26 \alpha) - N(1 + 151 \alpha) - 1001 \alpha - 1}{3N^{2}(N - 10 \alpha)} < 1 \]

if \( 1 < \alpha < \frac{2N^{3} - N^{2} + N + 1}{N^{2} + 156N^{2} - 151N - 1001} \) for \( N \geq 54 \). By Theorem 3.2, boundary value problem (4.1), (4.2) has a nontrivial solution \( x^{*} \) with \( \|x^{*}\|_{1} \leq \frac{13(1+\alpha)}{2N^{2} - \alpha} \) if \( N \in \mathbb{Z} \) is near infinity. □

References