Global stability for nonlinear dynamic equations with variable coefficients

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We give sufficient conditions under which the trivial solution of a nonlinear dynamic equation with variable coefficients is globally asymptotically stable, for arbitrary time scales unbounded above.

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1. Introduction

The unification and extension of continuous calculus, discrete calculus, q-calculus, and indeed arbitrary real-number calculus to time-scale calculus, where a time scale is simply any nonempty closed set of real numbers, were first accomplished by Hilger in [7]. Since then, time-scale calculus has made steady inroads in explaining the interconnections that exist among the various calculuses, and in extending our understanding to a new, more general and overarching theory. The purpose of this work is to illustrate this new understanding by extending some continuous and discrete delay equations to certain time scales.

Throughout this work the assumption is made that the time scale $T$ (any nonempty closed set of real numbers) is unbounded above and has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also assume throughout that $t_0 < t_1$ are points in $T$, and define the time-scale interval $[t_0, t_1]_T = \{ t \in T: t_0 \leq t \leq t_1 \}$. Other time-scale intervals are defined similarly. The notation $x^\Delta(t)$ is the delta derivative of $x: T \to \mathbb{R}$ at $t \in T$. It is defined by the relation

$$x^\Delta(t) := \lim_{s \to t} \frac{x(\sigma(t)) - x(s)}{\sigma(t) - s},$$

where $\sigma(t) := \inf \{ s \in T: s > t \}$ is the forward jump operator. We also define $\mu(t) := \sigma(t) - t$ to be the forward graininess function, $\rho(t) := \sup \{ s \in T: s < t \}$ to be the backward jump operator, and $\nu(t) := t - \rho(t)$ to be the backward graininess function. In particular, if $T = \mathbb{R}$, then $\sigma(t) = t$ and $x^\Delta = x'$, while if $T = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $x^\Delta = \Delta x$, where the symbol $\Delta$ is the usual forward difference operator. We often use $x^\Delta$ to represent the composition $x \circ \sigma$.

A function $f: T \to \mathbb{R}$ is right-dense continuous provided it is continuous at each right-dense point $t \in T$ (a point where $\sigma(t) = t$) and has a left-sided limit at each left-dense point $t \in T$ (a point where $\rho(t) = t$). The set of right-dense continuous
functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$. It can be shown that any right-dense continuous function $f$ has an antiderivative (a function $F : \mathbb{T} \to \mathbb{R}$ with the property $F' = f$ for all $t \in \mathbb{T}$). Then the Cauchy integral of $f$ is defined by

$$\int_{t_0}^{t_1} f(t) \Delta t = F(t_1) - F(t_0),$$

where $F$ is an antiderivative of $f$ on $\mathbb{T}$. For more on time scales and time-scale notation, please see the fundamental text by Bohner and Peterson [5].

Now consider, on arbitrary time scales $\mathbb{T}$ unbounded above, the nonlinear delay dynamic equation

$$x^\Delta(t) = -\sum_{j=0}^{n} a_j(t) f_j(x(\tau_j(t))), \quad t \in [t_0, \infty)_\mathbb{T},$$

$$x(t) = \psi(t), \quad t \in [\tau_n(t_0), t_0]_\mathbb{T}, \quad \psi \in C_{rd}([\tau_n(t_0), t_0]_\mathbb{T}).$$

Here the delays $\tau_j : \mathbb{T} \to \mathbb{T}$, $j \in \{0, 1, \ldots, n\}$, are right-dense continuous strictly increasing functions unbounded above with $\tau_n(\mathbb{T}) = \mathbb{T}$ such that for some constant $M > 0$

$$\rho^\mathbb{B}(t) - M \leq \tau_n(t) < \tau_{n-1}(t) < \cdots < \tau_0(t) = t,$$

the functions $f_j : \mathbb{R} \to \mathbb{R}$ are continuous, the right-dense continuous functions $a_j : \mathbb{T} \to \mathbb{R}$ satisfy

$$a_j(t) \geq 0, \quad \sum_{j=0}^{n} a_j(t) > 0, \quad \int_{t_0}^{\infty} \sum_{j=0}^{n} a_j(t) \Delta t = \infty,$$

and at $t_0$ in (1.1) the delta derivative is from the right. Furthermore, we assume that there exists a strictly increasing continuous function $f$ on $(-\infty, \infty)$ such that

$$f(0) = 0, \quad 0 < \frac{f(x)}{f'(x)} \leq 1, \quad x \neq 0, \quad 0 \leq j \leq n, \quad \lim_{x \to -\infty} f(x) \text{ finite if } f(x) \neq x.$$  

Note that existence and uniqueness of a solution of (1.1) follow from standard arguments similar to those given when $\mathbb{T} = \mathbb{R}$, for example in Kuang [9], Eq. (1.1) is studied extensively by Muroya, Ishiwata, and Guglielmi [10] in the case when $\mathbb{T} = \mathbb{Z}$; indeed many of our techniques in this paper are motivated directly by those in [10]; please also see Tkachenko and Trofimchuk [11]. See also a related discussion in Anderson, Krueger, and Peterson [4, Section 4], where delay equations with delay $\tau$ of the form

$$x^\Delta(t) = -\sum_{i=1}^{n} f_i(t, x(s)) \Delta s, \quad t \in [t_0, \infty)_\mathbb{T}$$

are treated.

Some recent papers dealing with first-order delay dynamic equations on time scales include the following. In [6], the authors Čermák and Urbánek consider the asymptotic properties of the first-order linear delay dynamic equation

$$x^\Delta(t) = a(t)x(t) + b(t)x(\tau(t)), \quad t \in \mathbb{T}.$$  

Another pair of authors dealing with first-order linear delay dynamic equations is Wu and Zhou [12], with the simpler equation

$$x^\Delta(t) + p(t)x(t - \tau(t)) = 0, \quad t \in \mathbb{T}.$$  

Kaufmann and Raffoul [8] consider the nonlinear neutral dynamic equation with delay expressed as

$$x^\Delta(t) = -a(t)x^\sigma(t) + (Q(t, x(t), x(t - g(t))))^\Delta + G(t, x(t), x(t - g(t))), \quad t \in \mathbb{T}.$$  

The asymptotic behavior of solutions for the neutral dynamic equation on time scales given by

$$[x(t) - p(t)x(k(t))]^\Delta + q(t)x(\ell(t)) = 0, \quad t \in [t_0, \infty)_\mathbb{T}$$

is investigated in [1]. For a general first-order delay dynamic equation

$$x^\Delta(t) = F(t, x(\tau(t))), \quad t \in [t_0, \infty)_\mathbb{T}$$

considered by Anderson and Kenz [3], the global asymptotic behavior is discussed, given certain constraints on the nonlinearity $F$; these ideas are extended somewhat in [2] to the forced first-order delay dynamic equation

$$x^\Delta(t) = -p(t)f(x(\tau(t))) + r(t), \quad t \in [t_0, \infty)_\mathbb{T}.$$
2. Basic lemmas

In this section we present and prove some basic lemmas, setting the stage for the main results in the sequel.

**Lemma 2.1.** Let \( x \) be the solution of (1.1). If there exists an \( s \in [t_0, \infty)_\tau \) such that \( x(t) > 0 (< 0) \) for all \( t \in [s, \infty)_\tau \), then \( x \) is eventually decreasing (increasing), and \( \lim_{t \to -\infty} x(t) = 0 \).

**Proof.** Assume \( x > 0 \) eventually. Then by (1.2) and (1.3), respectively, for \( t \) large enough

\[
x^\Delta(t) \leq -\sum_{j=0}^{n} a_j(t) f(x(\tau_j(t))) \leq -\sum_{j=0}^{n} a_j(t) f(0) = 0,
\]

so that \( x \) is eventually decreasing. Define \( \alpha := \lim_{t \to -\infty} x(t) \). If \( \alpha > 0 \), then there exists a \( \tilde{t} \in [t_0, \infty)_\tau \) such that \( x(t) \geq \alpha \) for \( t \geq \tau_n(\tilde{t}) \). Using this together with (1.2) and (1.3), we have

\[
x^\Delta(t) \leq -\sum_{j=0}^{n} a_j(t) f(\alpha), \quad t \in [\tilde{t}, \infty)_\tau.
\]

Delta integrating from \( \tilde{t} \) to \( t \) we get that

\[
x(t) \leq x(\tilde{t}) - \left( \int_{\tilde{t}}^{t} \sum_{j=0}^{n} a_j(s) \Delta s \right) f(\alpha),
\]

implying by (1.2) that \( \lim_{t \to -\infty} x(t) = -\infty \), a contradiction of \( \alpha > 0 \). Therefore, \( \lim_{t \to -\infty} x(t) = 0 \). The case for \( x < 0 \) eventually is similar and thus omitted. \( \square \)

**Lemma 2.2.** Let \( x \) be the solution of (1.1). If \( f(x) \neq x \) and

\[
\sup_{t \geq \tau_n^{-1}(t_0)} \int_{\tau_n(t)}^{\sigma(t)} \sum_{j=0}^{n} a_j(s) \Delta s < \infty,
\]

then \( x \) is bounded above and below.

**Proof.** Consider the case \( f(x) \neq x \), with

\[
\lim_{x \to -\infty} f(x) = -\beta > -\infty.
\]

From (1.1)–(1.3),

\[
x^\Delta(t) \leq \beta \sum_{j=0}^{n} a_j(t), \quad t \in [t_0, \infty)_\tau. \tag{2.1}
\]

Suppose \( \lim \sup_{t \to -\infty} x(t) = +\infty \). Then there exists a strictly monotone increasing time-scale sequence \( \{\tilde{t}_k\}_{k=1}^{\infty} \) in \([t_0, \infty)_\tau\) such that \( \lim_{k \to -\infty} \tilde{t}_k = \infty \), \( x^\sigma(\tilde{t}_k) \to \infty \) as \( k \to \infty \), \( x^\sigma(\tilde{t}_k) = \max_{0 \leq t \leq \sigma(\tilde{t}_k)} x(t) > 0 \) and \( x^\Delta(\tilde{t}_k) \geq 0 \). Then by (1.1),

\[
0 \leq x^\Delta(\tilde{t}_k) = -\sum_{j=0}^{n} a_j(\tilde{t}_k) f_j(x(\tau_j(\tilde{t}_k)))
\]

implying \( \sum_{j=0}^{n} a_j(\tilde{t}_k) f_j(x(\tau_j(\tilde{t}_k))) \leq 0 \). Therefore there exists \( \xi_k \in [\tau_n(\tilde{t}_k), \sigma(\tilde{t}_k)]_\tau \) such that \( x(\xi_k) \leq 0 \). Delta integrating (2.1) from \( \xi_k \) to \( \sigma(\tilde{t}_k) \) yields

\[
x^\sigma(\tilde{t}_k) \leq x(\xi_k) + \beta \int_{\xi_k}^{\sigma(\tilde{t}_k)} \sum_{j=0}^{n} a_j(t) \Delta t \leq \beta \lambda,
\]

where

\[
\sup_{t \geq \tau_n^{-1}(t_0)} \int_{\tau_n(t)}^{\sigma(t)} \sum_{j=0}^{n} a_j(s) \Delta s \leq \lambda < \infty.
\]
As a result, lim sup \( k \to \infty \) \( x^{(k)}(t_k) \leq \beta \lambda \), a contradiction. Thus, \( x(t) \leq \beta \lambda \) for all \( t \in [t_0, \infty) \). From (1.2) and (1.3), respectively,
\[
x^{(k)}(t) \geq -\sum_{j=0}^{n} a_j(t) f(\beta \lambda), \quad t \in [t_0, \infty).
\] (2.2)

Next, we will show that \( x \) is bounded below. Assume \( \liminf_{k \to \infty} x(t) = -\infty \). Then there exists a strictly monotone increasing time-scale sequence \( \{t_k\}_{k=1}^{\infty} \) in \( [t_0, \infty) \) such that \( \lim_{k \to \infty} t_k = \infty \), \( x^{(k)}(t_k) \to -\infty \) as \( k \to \infty \), \( x^{(k)}(t_k) = \min_{0 \leq t \leq \sigma(t_k)} x(t) < 0 \) and \( x^{(k)}(t_k) \leq 0 \). Then by (1.1),
\[
0 \geq x^{(k)}(t_k) = -\sum_{j=0}^{n} a_j(t_k) f(x(\sigma(t_k)));
\]

implying there exists \( \eta_k \in [\tau_n(t_k), \sigma(t_k)] \) such that \( x(\eta_k) \geq 0 \). Delta integrating (2.2) from \( \eta_k \) to \( \sigma(t_k) \) yields
\[
x^{(k)}(t) \geq x(\eta_k) - f(\beta \lambda) \int_{\eta_k}^{\sigma(t_k)} \sum_{j=0}^{n} a_j(t) \Delta t \geq -\lambda f(\beta \lambda).
\]

As a result, \( \liminf_{k \to \infty} x^{(k)}(t_k) \geq -\lambda f(\beta \lambda) \), a contradiction. Thus \( x \) is bounded below as well. \( \square \)

Lemma 2.3. Let \( x \) be the solution of (1.1). If there exists a point \( s \in [\tau_{n-1}(t_0), \infty) \) such that \( x^{(s)}(s) > 0 \) and \( x^{(s)}(s) > 0 \), then there exists a \( g(s) \in [\tau_n(s), \sigma(s)] \) such that
\[
x(g(s)) = \min_{t \in [\tau_n(s), \sigma(s)]} x(t) < 0.
\] (2.3)

If there exists a point \( s \in [\tau_{n-1}(t_0), \infty) \) such that \( x^{(s)}(s) < 0 \) and \( x^{(s)}(s) < 0 \), then there exists a \( g(s) \in [\tau_n(s), \sigma(s)] \) such that
\[
x(g(s)) = \max_{t \in [\tau_n(s), \sigma(s)]} x(t) > 0.
\] (2.4)

Proof. If \( x(t) > 0 \) for all \( t \in [\tau_n(s), \sigma(s)] \), then by (1.2) and (1.3) we have \( x^{(s)}(s) \leq 0 \), a contradiction. On the other hand, if \( x(t) < 0 \) for all \( t \in [\tau_n(s), \sigma(s)] \), then again by (1.2) and (1.3) we have \( x^{(s)}(s) \geq 0 \), another contradiction. \( \square \)

Remark 2.4. Let
\[
\begin{align*}
1 &= \sup_{s \geq \tau_{n-1}(t_0)} \sum_{j=0}^{n} \int_{\tau_{n-1}^{1}(t_0)}^{\tau_{n}^{1}(t_0)} a_j(t) \Delta t, \\
2 &= \sup_{s \geq \tau_{n-1}(t_0)} \sum_{j=1}^{n} \int_{\tau_{n}^{1}(t_0)}^{\sigma(s)} a_j(t) \Delta t, \\
\psi(x) &= x - f(L).
\end{align*}
\] (2.5)

If \( f(x) \neq x \) and \( r_1 + r_2 < \infty \), then by Lemma 2.2 we see that any solution \( x \) of (1.1) that oscillates about 0 is bounded above and below.

For future reference, given any real number \( L < 0 \), use (2.5) to define
\[
R_L := \max_{L \leq x \leq 0} \psi(x) - r_2 f(L) \quad \text{and} \quad S_L := \min_{0 \leq x \leq |R_L|} \psi(x) - r_2 f(R_L).
\] (2.6)

Lemma 2.5. Suppose the solution \( x \) of (1.1) oscillates about 0. If for some real number \( L < 0 \) there exists a point \( t_L \in [\tau_{n-2}(t_0), \infty] \) such that \( x(t_L) \geq L \) for \( t \in [T_2, \infty) \), then in (2.6) we have \( R_L > 0 \) and \( S_L < 0 \), and the solution \( x \) satisfies the inequalities
\[
x^{(i)}(t) \leq R_L \quad \text{for} \quad t \in [T_2, \infty) \quad \text{and} \quad x^{(i)}(t) \geq S_L \quad \text{for} \quad t \in [T_3, \infty),
\] (2.7)

where \( T_2 \geq \tau_{n-2}(t_1) \) and \( T_3 \geq \tau_{n-4}(t_2) \) are such that there exists a point \( s \in [T_i, \infty) \) such that \( (-1)^i x^{(i)}(s) > 0 \) and \( (-1)^i x^{(i)}(s) > 0 \), \( i \in [2, 3] \). Furthermore, if \( S_L > L \) for any \( L < 0 \), then \( \lim_{t \to \infty} x(t) = 0 \).

Proof. Assume \( x(t) \geq L \) for any \( t \in [T_1, \infty) \). By the oscillation assumption, there exists a point \( s \in [T_1, \infty) \) such that \( x^{(i)}(s) > 0 \) and \( x^{(i)}(s) > 0 \). Then by Lemma 2.3, there exists a point \( g(s) \in [\tau_n(s), \sigma(s)] \) such that
\[
x(g(s)) = \min_{t \in [\tau_n(s), \sigma(s)]} x(t) < 0.
\]
Using (1.1), delta integrate from \( g(s) \) to \( \sigma(s) \) to obtain
\[
0 < x^{\sigma}(s) = x(g(s)) - \int_{g(s)}^{\sigma(s)} \sum_{j=0}^{n} a_j(t_j f_j(x(t_j))) \Delta t.
\] (2.8)

Since \( \tau_0 > T_3 \), using (1.2) and (1.3) we know there exists a finite collection of points \( \{ \tau_j(s) \}_{j=0}^{n} \) from \([g(s), \sigma(s)]_T \) with \( \tau_0 = g(s) \) and
\[
\tau_j(s) = \begin{cases} \tau_j^{-1}(\tau_0(s)), & \tau_j^{-1}(\tau_0(s)) > g(s), \\ g(s), & \text{otherwise}. \end{cases}
\]

such that \( \tau_j(t_j(t)) \in [\tau_0, \sigma(s)]_T \) and \( f_j(x(\tau_j(t))) > f(\sigma(x(s))) \) for all \( t \in [\tau_j(s), \sigma(s)]_T \), and for all \( t \in [g(s), \sigma(s)]_T \) we have that \( f_j(x(\tau_j(t))) > f(L) \). Thus we can rewrite (2.8) as the inequality
\[
x^{\sigma}(s) \leq x(g(s)) - \sum_{j=0}^{n} \left( \int_{g(s)}^{\tau_j(s)} a_j(t_j(t)) \Delta t \right) f(x(g(s))) - \sum_{j=1}^{n} \left( \int_{g(s)}^{\tau_j(s)} a_j(t_j(t)) \Delta t \right) f(\sigma(x(s))) - r_2 f(L).
\]

Note in (2.5) that \( r_1 \) and \( r_2 \) are independent of \( g(s) \) and the choice of the \( \tau_j \) earlier in this particular proof. Using (2.6) and (2.5), we have the inequality
\[
0 < x^{\sigma}(s) \leq \psi(x(g(s))) - r_2 f(L) \leq R_L.
\]

If there exists a time-scale point \( t \geq T_3 \) such that \( x^{\sigma}(t) > R_L \), then by the given assumptions, there exists a time-scale point \( s \geq T_3 \) such that \( x^{\sigma}(s) > 0 \), \( x^4(s) > 0 \), and \( x^{\sigma}(s) > R_L \), a contradiction. Thus, we have that \( x^{\sigma}(t) \leq R_L \) for any \( t \in [T_2, \infty)_T \).

Similarly, there exists a time-scale point \( s \in [T_1, \infty)_T \) such that \( x^{\sigma}(s) < 0 \) and \( x^4(s) < 0 \). Then by Lemma 2.3, there exists a point \( \bar{g}(s) \in [\tau_0, \sigma(s)]_T \) such that
\[
x(\bar{g}(s)) = \max_{t \in [T_0, \sigma(s)]_T} x(t) > 0.
\]

and, just as earlier in the proof for the previous case,
\[
0 > x^{\sigma}(s) \geq x(\bar{g}(s)) - \sum_{j=0}^{n} \left( \int_{g(s)}^{\tau_j(s)} a_j(t_j(t)) \Delta t \right) f(\bar{g}(s)) - \sum_{j=1}^{n} \left( \int_{g(s)}^{\tau_j(s)} a_j(t_j(t)) \Delta t \right) f(R_L) \geq S_L.
\]

If there exists a time-scale point \( t \geq T_3 \) such that \( x^{\sigma}(t) < S_L \), then by the given assumptions, there exists a time-scale point \( s \geq T_2 \) such that \( x^{\sigma}(s) < 0 \), \( x^4(s) < 0 \), and \( x^{\sigma}(s) < S_L \), that is, (2.7).

Moreover, suppose that \( S_L > L \) for any \( L < 0 \), and set \( L := \liminf_{t \to \infty} x(t) \). If \( L < 0 \), then we have that \( S_L > L \). As a result, there would exist a time-scale point \( \tilde{t} \geq \tau_0^{-1}(L) \) such that \( x(t) \geq S_L > L \) for any \( t \in [T_2, \infty)_T \), a contradiction. Therefore, we have that \( L = 0 \), so that \( \lim_{t \to \infty} x(t) = 0 \). \( \square \)

3. Main results

In this section we present our main asymptotic results concerning Eq. (1.1). First, we need the following definitions.

Definition 3.1. The zero solution of (1.1) is uniformly stable if and only if for any \( \epsilon > 0 \) and any \( t^* \in [T_0, \infty)_T \), there exists a \( \delta = \delta(\epsilon) > 0 \) such that
\[
\sup_{x \in \tau_{0}, \sigma(t^*)} |x(s)| < \delta
\]
implies that the solution \( x(t) \) satisfies \( |x(t)| < \epsilon \) for \( t \in [t^*, \infty)_T \).

Definition 3.2. The zero solution of (1.1) is globally attractive if and only if every solution of (1.1) tends to zero as \( t \to \infty \) in the time scale.

Definition 3.3. The zero solution of (1.1) is globally asymptotically stable if and only if it is uniformly stable and globally attractive.
Remark 3.4. Assume \( f(x) \neq x \),

\[
\sup_{t \geq t_L^{-1}(t_0)} \int_{t_0}^{t} \sum_{j=0}^{n} a_j(s) \Delta s < \infty.
\]

and that the hypothesis of Lemma 2.1, or the hypotheses of Lemmas 2.2 and 2.5, hold. Then we have that the zero solution of (1.1) is uniformly stable. As a result, \( \lim_{t \to \infty} x(t) = 0 \) implies that the zero solution of (1.1) is globally asymptotically stable.

**Proof.** This sketch is motivated by ideas from Muroya, Ishiwata, and Guglielmi [10]. Assume that the solution \( x \) of (1.1) is greater (less) than 0. Then, by Lemma 2.1, we have that \( x \) is decreasing (increasing) to zero, and \( \lim_{t \to \infty} x(t) = 0 \). This in turn implies that if for any \( \epsilon > 0 \) and time-scale point \( T \geq t_0 \), there is a \( \delta = \epsilon > 0 \) such that \( \sup \{|x(t_j(T))|: 0 \leq j \leq n| < \delta \} \), then the solution \( x \) of (1.1) satisfies \( |x(t)| < \epsilon \) for \( t \in [T, \infty) \), by the property of the solution which converges monotonically to the zero solution of (1.1). The case where \( x \) is eventually positive (negative) is similar.

Next, suppose that the solution \( x \) of (1.1) is oscillatory about 0. Then, by Lemmas 2.2 and 2.5 we have that if for any \( \epsilon > 0 \) and time-scale point \( T \geq t_0 \), there is a \( \delta > 0 \) such that \( \epsilon = \max(\delta, R_L) > 0 \) and \( \sup \{|x(t_j(T))|: 0 \leq j \leq n| < \delta \} \), then by the time-scale induction principle [5, Theorem 1.7], for \( L = -\delta < 0 \), the solution \( x \) of (1.1) satisfies \( -\epsilon \leq -L \leq x(t) \leq R_L \leq \epsilon \) for \( t \in [T, \infty) \) by Lemma 2.5.

Thus the solution \( x \) of (1.1) satisfies Definition 3.1. \( \square \)

The main results of the paper are now presented. Because the foundational lemmas from the discrete case proven by Muroya, Ishiwata, and Guglielmi [10] have all been generalized to arbitrary time scales in the previous section and in the remark above, the statements and proofs of the following theorems are very similar to those found in the discrete case in [10].

**Remark 3.5.** The following theorems articulate hypotheses that guarantee that \( S_L > L \) for all \( L < 0 \), where \( S_L \) is given in (2.6).

**Theorem 3.6.** Assume \( f(x) \neq x \) and

\[
\sup_{t \geq t_L^{-1}(t_0)} \int_{t_0}^{t} a_j(s) \Delta s < \infty.
\]

Moreover, assume that \( \varphi \) defined in (2.5) is monotone on \((0, \infty)\), and that for any \( L < 0 \)

\[
\begin{cases}
-2L \varphi(-2L) > L & \text{if } \varphi \text{ is increasing}, \\
\varphi(\varphi(L) - 2L) - 2L \varphi(\varphi(L) - 2L) > L & \text{if } \varphi \text{ is decreasing},
\end{cases}
\]

(3.1)

for \( R_2 \) given in (2.5). Then the zero solution of (1.1) is globally asymptotically stable.

**Proof.** By Lemma 2.1, it is enough to consider the case where the solution \( x \) of (1.1) is oscillatory about 0. If for some real number \( L < 0 \) there exists a time-scale point \( t_0 \in \mathbb{T} \), \( \infty_T \) such that \( x(t) \geq L \) for \( t \in [t_L, \infty)_T \), then for any \( t \in [\tau_n^{-4}(t_L), \infty)_T \) we have

\[
x^L(t) \leq R_L \text{ for } t \in [\tau_n^{-2}(t_L), \infty)_T, \quad \text{and} \quad x^L(t) \geq S_L \text{ for } t \in [\tau_n^{-4}(t_L), \infty)_T,
\]

where \( R_L \) and \( S_L \) are defined as in (2.6). Assume that \( \varphi \) is monotone increasing on \((0, \infty)\). Then \( \max_{0 \leq x \leq R_L} \varphi(x) = 0 \) and \( \min_{0 \leq x \leq R_L} \varphi(x) = 0 \). Consequently for (2.6) in Lemma 2.5 we have \( R_L = -r_2f(L) \) and \( S_L = -r_2f(R_L) \). Thus by (3.1) we have \( S_L > L \). Applying Lemma 2.5, we see that \( \lim_{t \to \infty} x(t) = 0 \), and by Remark 3.4, the zero solution of (1.1) is globally asymptotically stable. The case where \( \varphi \) is monotone decreasing on \((0, \infty)\) is omitted due to its similarity. \( \square \)

**Example 3.7.** Let \( \mathbb{T} \) be an isolated time scale unbounded above with \( t_0 \in \mathbb{T} \). If \( \sup_{\rho \geq \sigma(t_0)} \mu(s) = \sup_{\rho \geq \sigma(t_0)} \nu(s) = 1 \), then the zero solution of

\[
\begin{cases}
x^L(t) = -a_1 \arctan(x(\rho(t)) \quad t \in [t_0, \infty)_T, \\
x(t) = \psi(t) \quad t \in \{\rho(t_0), t_0\}, \quad \psi \in C_{\rho_0}(\{\rho(t_0), t_0\}),
\end{cases}
\]
is globally asymptotically stable for any constant \( a_1 \in (0, 1] \).

**Proof.** Let \( t_0 \in \mathbb{T}, a_0 = 0 \), \( a_1 \in (0, 1] \) be constant, and \( \tau_1 = \rho(t) \). By (2.5),
Theorem 3.9. Then the zero solution of (1.1) is globally asymptotically stable.

Example 3.10. Let \( T \) be an isolated time scale unbounded above with \( t_0 \in T \). If \( \sup_{s \geq \sigma(t_0)} \mu(s) = \sup_{s \geq \sigma(t_0)} v(s) = 1 \), then the zero solution of

\[
\begin{aligned}
x^{\hat{a}}(t) &= -a_1(e^{(a_1)} - 1), \quad t \in [t_0, \infty)_T, \\
x(t) &= \psi(t), \quad t \in \{\rho(t_0), t_0\}, \quad \psi \in C_{\text{rd}}(\{\rho(t_0), t_0\}),
\end{aligned}
\]

is globally asymptotically stable for any constant \( a_1 \in (\ln2, 1] \).

**Proof.** Let \( t_0 \in T, a_0 = 0, a_1 \in (\ln2, 1] \) be constant, and \( r_1(t) = \rho(t) \). By (2.5),

\[
\begin{aligned}
r_1 &= \sup_{s \geq \sigma(t_0)} \int_{\sigma(s)}^{s} a_1 \Delta t = a_1 \cdot \sup_{s \geq \sigma(t_0)} \mu(s) = a_1 \leq 1, \\
r_2 &= \sup_{s \geq \sigma(t_0)} \int_{\rho(s)}^{s} a_1 \Delta t = a_1 \cdot \sup_{s \geq \sigma(t_0)} v(s) = a_1 \leq 1,
\end{aligned}
\]

and \( \varphi(x) = x - a_1 \arctan(x) \). Then \( \varphi \) is increasing, and

\[
-a_1 \arctan(-a_1 \arctan(L)) > L, \quad \forall L < 0.
\]

Thus by Theorem 3.8, the zero solution of (1.1) is globally asymptotically stable. \( \square \)

**Theorem 3.8.** Assume \( f(x) \neq x \) and

\[
\sup_{t \geq t_0} \int_{t_0}^{t} \sum_{j=0}^{n} a_j(s) \Delta s < \infty.
\]

Moreover, assume that \( \varphi \) is unimodal with a global maximum at \( R^* = 0 \), and that for any \( L < 0 \)

\[
\varphi(-r_2 f(L)) - r_2 f(-r_2 f(L)) > L.
\]

Then the zero solution of (1.1) is globally asymptotically stable.

**Theorem 3.9.** Assume \( f(x) \neq x \) and

\[
\sup_{t \geq t_0} \int_{t_0}^{t} \sum_{j=0}^{n} a_j(s) \Delta s < \infty.
\]

Moreover, assume that \( \varphi \) is unimodal with a global maximum at \( R^* > 0 \), and that for any \( L < 0 \)

\[
\begin{aligned}
\varphi(-r_2 f(L)) - r_2 f(-r_2 f(L)) &> L \quad \text{if } L \leq \hat{L}, \\
-r_2 f(-r_2 f(L)) &> L \quad \text{if } -\infty \leq L < 0,
\end{aligned}
\]

where if \( \varphi(x) > 0 \) for any \( x > 0 \) then \( \hat{L} = -\infty \), otherwise \( \hat{L} < 0 \) is uniquely defined by \( \varphi(-r_2 f(\hat{L})) = 0 \). Then the zero solution of (1.1) is globally asymptotically stable.
Theorem 3.11. Assume \( f(x) \neq x \) and
\[
\sup_{j \geq 1} \int_{t_j(0)}^{t_j(T)} \sum_{j=0}^{n} a_j(t) \, ds < \infty.
\]

Moreover, assume that \( \varphi \) is unimodal with a global maximum at \( L^* < 0 \), and that
\[
\mathcal{T}(L) > L, \quad \text{for any } L < 0,
\]
(3.4)

where
\[
\mathcal{T}(L) := \varphi(\max[L^*, L]) - r_2 f(L) - r_2 f(\max[L^*, L]) - r_2 f(L).
\]

Then the zero solution of (1.1) is globally asymptotically stable.

Proof. For the proofs of Theorems 3.8, 3.9, and 3.11, use the results in Lemma 2.3 and apply Lemma 2.5 for each case in lines (3.2)–(3.4), respectively. \( \square \)

Remark 3.12. Let \( T = \mathbb{Z} \), the set of integers. To compare with the results in [10], take \( t_0 = 0 \) and \( t_j(s) := s - j \). Then (2.5) becomes
\[
\begin{aligned}
& r_1 := \sup_{m \geq n} \sum_{j=0}^{m} a_j(t), \\
& r_2 := \sup_{m \geq n} \sum_{j=0}^{m} a_j(t).
\end{aligned}
\]

If we switch the order of summation and reindex we get
\[
\begin{aligned}
& r_1 := \sup_{m \geq n} \sum_{k=0}^{n-k} \sum_{j=0}^{m} a_j(m-k), \\
& r_2 := \sup_{m \geq n} \sum_{k=1}^{n} \sum_{j=n-k+1}^{m} a_j(m-k),
\end{aligned}
\]
that is exactly the form given in [10, Line 1.10].

4. Special case where \( f(x) = x \)

In this section we consider the special case where \( f(x) = x \) in (1.1).

Theorem 4.1. Assume in (1.1) that \( f(x) = x \), and that
\[
\begin{aligned}
& r_1 + r_2 < \frac{3}{2} + \frac{1}{2} \sup [\mu(t): t \in T] \quad \text{or} \quad r_2 < 1 \quad \text{if } r_1 \leq 1, \\
& r_1 + r_2 < \frac{1}{2} \inf [\sigma(t) - \sigma(0): t \in T] \quad \text{if } r_1 > 1.
\end{aligned}
\]

(4.1)

Then the zero solution of (1.1) is globally asymptotically stable.

Proof. We provide a sketch of the proof. Since \( f(x) = x \), it can be shown that if
\[
r_1 + r_2 < \frac{3}{2} + \frac{1}{2} \sup [\sigma(t) - \sigma(0): t \in T],
\]
then the zero solution of (1.1) is globally asymptotically stable; this can be done by extending the proof of [11, Theorem 1.3] in the discrete case to general time scales using the method in [3, Theorem 3.1]. Further, the function \( \varphi(x) = x(1 - r_1) \) is monotone on \((-\infty, \infty)\), and (3.1) in this case says that for any \( L < 0 \),
\[
(-r_2)^2 L > L \quad \text{if } r_1 \leq 1, \quad (1 - (r_1 + r_2)^2) L > L \quad \text{if } r_1 > 1,
\]
which is equivalent to (4.1). Next we show that if \( x \) is oscillatory about 0 and (4.1) holds, then \( x \) is bounded above and below. Assume that \( \limsup_{t \to \infty} |x(t)| = \infty \). Then there exists a strictly monotone increasing time-scale sequence \( \{t_k\}_{k=1}^\infty \) in \([t_0, \infty)_T\) such that \( \lim_{k \to \infty} t_k = \infty \), \( |x^\sigma(t_k)| \to \infty \) as \( k \to \infty \), and \( |x^\sigma(t_k)| \geq |x(t)| \) for any \( t \in [t_0, \sigma(t_k)]_T \). If \( x^\sigma(t_k) > 0 \), then similar to
the proofs of Lemmas 2.2 and 2.5 we have that there exists \( \xi_k \in [\tau_n(t_k), \sigma(t_k)] \) such that \( x(\xi_k) \leq 0 \), \( x(t) \geq L = -x(\sigma(t_k)) \) for \( t \leq t_k \), and

\[
x^p(t_k) \leq x(\xi_k) - \sum_{j=0}^{n} \left( \int_{\tau^{-1}(\tau_n(t_k))}^{\sigma(t_k)} \frac{a_j(t)}{\Delta t} f\left(x(\xi_k)\right) - \sum_{j=1}^{n} \left( \int_{\tau_n(s)}^{\tau_n(t_k)} \frac{a_j(t)}{\Delta t} f(L) \right) \right)
\]

\[
\leq \varphi(x(\xi_k)) - r_2 f(L) \leq R_1.
\]

If \( r_1 \leq 1 \) and \( r_2 < 1 \), then \( R_1 = -r_2 L < -L \), and if \( r_1 > 1 \) and \( r_1 + r_2 < 2 \), then \( R_1 = L(1 - r_1) - r_2 L = [(r_1 + r_2) - 1](-L) < -L \), a contradiction. For the case \( x^p(t_k) < 0 \), we similarly arrive at a contradiction. Therefore \( x \) is bounded above and below.

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References