Existence of solutions for a first-order $p$-Laplacian BVP on time scales

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Abstract

We introduce a first-order delta dynamic equation on time scales involving the one-dimensional $p$-Laplacian, and prove the existence of at least one positive solution. An example applying our result is also given.

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1. Introduction

We study the first-order delta dynamic equation on time scales with the one-dimensional $p$-Laplacian given by

$$
\varphi_p \left( u^\Delta(t) \right) = h(t) f \left( u^\sigma(t) \right), \quad t \in [a, b]_T,
$$

with boundary condition

$$
u(a) = B_0 \left( u^\Delta(b) \right),
$$

where $T$ is a time scale, $B_0 : \mathbb{R} \to \mathbb{R}$ is continuous, $\varphi_p(x) = |x|^{p-2}x$ for $p > 1$ is the one-dimensional $p$-Laplacian, and $(\varphi_p)^{-1} = \varphi_q$ with $1/p + 1/q = 1$. Here the function $f : [0, \infty) \to [0, \infty)$ is continuous and $h : [a, b]_T \to [0, \infty)$ is right-dense continuous such that the product $h(t)f(u^\sigma(t))$ is nonincreasing for $t \in [a, b]_T$ for nondecreasing functions $u$, with $hf$ not identically zero. Moreover, we assume that there exist constants $B_1, B_2$ satisfying $0 \leq B_1 \leq B_2$ such that

$$B_1 x \leq B_0(x) \leq B_2 x, \quad x \geq 0.
$$

Let

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{\varphi_p(x)} \quad \text{and} \quad f_\infty = \lim_{x \to \infty} \frac{f(x)}{\varphi_p(x)}.
$$

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In the case of the time scales $\mathbb{R}$ (the real numbers), or $h\mathbb{Z}$ (a constant graininess), the $p$-Laplacian arises in non-Newtonian fluids, in some reaction–diffusion problems, in flow through porous media, in nonlinear elasticity, glaciology and petroleum extraction; for a few references to such applications, see [1–7].

In Section 2, some cone preliminaries are presented. Then, in Section 3, in the case where $f$ is sublinear with respect to $\varphi_p$, we apply the Krasnosel’skii fixed point theorem and obtain the existence of at least one positive solution of (1.1) and (1.2). There has been much recent interest in second-order delta dynamic equations on time scales involving the one-dimensional $p$-Laplacian [8–24], but less on first-order ones.

Throughout this work we assume a knowledge of time scales and time-scale notation, first introduced by Hilger [25]. For more on time scales, please see the texts by Bohner and Peterson [26,27].

2. Preliminaries

In this section, we provide some background material to facilitate analysis of problem (1.1) and (1.2). First notice that by (1.1),

$$u^\Delta(t) = \varphi_q \left( h(t) f \left( u^\sigma(t) \right) \right) \geq 0, \quad t \in [a, b]_T,$$

with boundary condition

$$B_0 \left( u^\Delta(b) \right) = B_0 \left( \varphi_q \left( h(b) f \left( u^\sigma(b) \right) \right) \right) = u(a).$$

Delta integrating (2.1) from $a$ to $t$ and using the previous line we arrive at

$$u(t) = B_0 \left( \varphi_q \left( h(b) f \left( u^\sigma(b) \right) \right) \right) + \int_{a}^{t} \varphi_q \left( h(s) f \left( u^\sigma(s) \right) \right) \Delta s.$$

Let $E = C_0[a, \sigma(b)]_T$ be the Banach space with the sup-norm $\|u\| = \text{sup}\{|u(s)| : s \in [a, \sigma(b)]_T\}$, and define the cone $P \subset E$ by

$$P = \{u \in E : u(t) \geq 0, t \in [a, b]_T, u^\Delta \text{ nonincreasing}\}.$$ 

Clearly, $\|u\| = u^\sigma(b)$ for $u \in P$. For $u \in P$ define the operator $L$ by

$$(Lu)^\Delta(t) = \varphi_q \left( h(t) f \left( u^\sigma(t) \right) \right) \geq 0, \quad t \in [a, b]_T.$$ 

As $\varphi_q$ is nondecreasing and the product $h(t) f(u^\sigma(t))$ is nonincreasing for $u \in P$, $(Lu)^\Delta$ is nonincreasing. Therefore $L : P \to P$, and $\|Lu\| = (Lu)^\sigma(b)$. Moreover, $L$ is a completely continuous operator by a standard application of the Arzela–Ascoli theorem.

**Lemma 2.1.** For any $u \in P$ and $t \in [a, \sigma(b)]_T$ we have $u(t) \geq \frac{t-a}{\sigma(b)-a} \|u\|$. 

**Proof.** Let $u \in P$, so that $u(t), u^\Delta(t) \geq 0$ and $u^\Delta(t)$ is nonincreasing on $[a, b]_T$. Consequently for $t \in [a, \sigma(b)]_T$,

$$u(t) - u(a) = \int_{a}^{t} u^\Delta(\tau) \Delta \tau \geq (t-a)u^\Delta(t)$$

and

$$u^\sigma(b) - u(t) = \int_{t}^{\sigma(b)} u^\Delta(\tau) \Delta \tau \leq (\sigma(b)-t)u^\Delta(t).$$

From these two inequalities we obtain

$$u(t) \geq \frac{(\sigma(b)-t)u(a) + (t-a)u^\sigma(b)}{\sigma(b)-a} \geq \frac{t-a}{\sigma(b)-a} u^\sigma(b) = \frac{t-a}{\sigma(b)-a} \|u\|,$$

which completes the proof. □
3. Existence of positive solutions

We now prove our main theorem for the time-scale one-dimensional $p$-Laplacian boundary value problem. To establish an existence result we will employ the following fixed point theorem due to Krasnosel’skii [28], and seek a fixed point of $L$ in $P$.

**Theorem 3.1.** Let $E$ be a Banach space, $P \subseteq E$ be a cone, and suppose that $S_1$, $S_2$ are bounded open balls of $E$ centered at the origin with $S_1 \subset S_2$. Suppose further that $L : P \cap (\bar{S}_2 \setminus S_1) \to P$ is a completely continuous operator such that

(i) $\|Ly\| \leq \|y\|$, $y \in P \cap \partial S_1$ and $\|Ly\| \geq \|y\|$, $y \in P \cap \partial S_2$, or

(ii) $\|Ly\| \geq \|y\|$, $y \in P \cap \partial S_1$ and $\|Ly\| \leq \|y\|$, $y \in P \cap \partial S_2$

holds. Then $L$ has a fixed point in $P \cap (\bar{S}_2 \setminus S_1)$.

Applying Theorem 3.1 we get the following existence result.

**Theorem 3.2.** The boundary value problem (1.1) and (1.2) has at least one positive solution for $f_0 = \infty$ and $f_\infty = 0$, each given in (1.3).

**Proof.** Let $\gamma := \min\{t \in \mathbb{T} : \frac{a + \sigma(b)}{2} \leq t < b\}$ such that $(\gamma, b_\mathbb{T}) \neq \emptyset$. Since $f_0 = \infty$, there exists a constant $k > 0$ such that $f(x) \geq \varphi_p(c)\varphi_p(x) = \varphi_p(cx)$ for $0 < x \leq k$, where the number $c$ is chosen so that

$$c \left[ B_1\varphi_q(h(b)) + \frac{\gamma - a}{\sigma(b) - a} \int_{\gamma}^{\sigma(b)} \varphi_q(h(s)) \Delta s \right] \geq 1. \quad (3.1)$$

If $u \in P$ with $\|u\| = u^\sigma(b) = k$, then for $L$ given in (2.2) we have

$$\|Lu\| = Lu(\sigma(b)) = B_0(\varphi_q(h(b)f(u^\sigma(b)))) + \int_{a}^{\sigma(b)} \varphi_q(h(s)f(u^\sigma(s))) \Delta s$$

$$\geq B_1\varphi_q(h(b)\varphi_p(cu^\sigma(b))) + \int_{\gamma}^{\sigma(b)} \varphi_q(h(s)\varphi_p(cu^\sigma(s))) \Delta s$$

$$= B_1cu^\sigma(b)\varphi_q(h(b)) + c\int_{\gamma}^{\sigma(b)} u^\sigma(s)\varphi_q(h(s)) \Delta s$$

$$\geq B_1cu^\sigma(b)\varphi_q(h(b)) + c\frac{\gamma - a}{\sigma(b) - a} \|u\| \int_{\gamma}^{\sigma(b)} \varphi_q(h(s)) \Delta s \geq \|u\|.$$

Consequently, if we take $S_k = \{u \in E : \|u\| < k\}$, then $\|Lu\| \geq \|u\|$ for all $u \in P \cap \partial S_k$. Next consider $f_\infty = 0$. By definition there exists an $m' > 0$ such that $f(u) \leq \varphi_p(d)\varphi_p(u) = \varphi_p(du)$ for all $u \geq m'$, where $d$ is such that

$$d \left[ B_2\varphi_q(h(b)) + \int_{a}^{\sigma(b)} \varphi_q(h(s)) \Delta s \right] \leq 1. \quad (3.2)$$

First, suppose $f$ is bounded. Then $f(u) \leq \varphi_p(M)$ for all $u \in [0, \infty)$, for some constant $M > 0$. Take

$$m = \max \left\{ 2k, M \left[ B_2\varphi_q(h(b)) + \int_{a}^{\sigma(b)} \varphi_q(h(s)) \Delta s \right] \right\}. \quad (3.3)$$

If $u \in P$ with $\|u\| = m$, then

$$\|Lu\| = Lu(\sigma(b)) = B_0(\varphi_q(h(b)f(u^\sigma(b)))) + \int_{a}^{\sigma(b)} \varphi_q(h(s)f(u^\sigma(s))) \Delta s$$

$$\leq B_2\varphi_q(h(b)\varphi_p(M)) + \int_{a}^{\sigma(b)} \varphi_q(h(s)\varphi_p(M)) \Delta s$$

$$= MB_2\varphi_q(h(b)) + M \int_{a}^{\sigma(b)} \varphi_q(h(s)) \Delta s \leq m = \|u\|$$
by the choice of \( m \). On the other hand, if \( f \) is unbounded, take \( m \geq \max\{2k, \frac{\sigma(b)-a}{\sigma(b)+a}m'\} \) so that \( f(u) \leq f(m) \) for \( 0 < u \leq m \). If \( u \in P \) with \( \|u\| = m \), then using (3.2) we obtain

\[
\|Lu\| = Lu(\sigma(b)) = B_0 \left( \varphi_0 \left( h(b) f \left( u(\sigma(b)) \right) \right) \right) + \sum_{\Delta} \varphi_0 \left( h(s) f \left( u(\sigma(s)) \right) \right) \Delta s \\
\leq B_2 \varphi_0 \left( h(b) f (m) \right) + \sum_{\Delta} \varphi_0 \left( h(s) f (m) \right) \Delta s \\
\leq dm B_2 \varphi_0 (h(b)) + dm \sum_{\Delta} \varphi_0 (h(s)) \Delta s \leq m = \|u\|.
\]

Thus it is shown that in either instance, if we take \( S_m = \{ u \in E : \|u\| < m \} \), then we have \( \|Lu\| \leq \|u\| \) for \( u \in P \cap \partial S_m \). It follows from Theorem 3.1 (ii) that \( L \) has a fixed point \( u \) in \( P \cap (\overline{S_m} \setminus S_k) \) with \( k \leq \|u\| \leq m \). □

4. Example

Let \( T = \mathbb{R} \), and consider the unit interval differential equation

\[
u' |u'| = (1 - t) (4 - \arctan(u)) \quad \text{for} \ 0 < t < 1, \quad (4.1)
\]

with boundary conditions

\[
u(0) = u'(1) + \sin (u'(1)). \quad (4.2)
\]

Here we have taken \( p = 3 \), \( h(t) = (1 - t) \), \( f(w) = 4 - \arctan w \), and \( B_0(x) = x + \sin x \). Then \( B_1 = 0.5 \) and \( B_2 = 2 \), so that \( B_1 x \leq B_0(x) \leq B_2 x \) for all \( x \geq 0 \). Note that \( hf \) is nonincreasing, with

\[
f_0 = \lim_{x \to 0^+} \frac{f(x)}{\varphi_0(x)} = \infty \quad \text{and} \quad f_\infty = \lim_{x \to \infty} \frac{f(x)}{\varphi_0(x)} = 0.
\]

If \( c = 6\sqrt{2} \), then (3.1) is satisfied, and we can take \( k = 0.2288 \). For the condition on \( m \) in (3.3), \( m = \max\{2k, 2\sqrt{2}/3\} = 4/3 \). Therefore, by Theorem 3.2, the boundary value problem (4.1) and (4.2) has a positive solution \( u \) such that \( 0.2288 \leq \|u\| \leq 4/3 \).

References

[3] M. Bertsch, D. Hilhorst, The interface between regions where \( c \) is satisfied, and we can take \( m \). On the other hand, if \( f \) is unbounded, take \( m \geq \max\{2k, \frac{\sigma(b)-a}{\sigma(b)+a}m'\} \) so that \( f(u) \leq f(m) \) for \( 0 < u \leq m \). If \( u \in P \) with \( \|u\| = m \), then using (3.2) we obtain

\[
\|Lu\| = Lu(\sigma(b)) = B_0 \left( \varphi_0 \left( h(b) f \left( u(\sigma(b)) \right) \right) \right) + \sum_{\Delta} \varphi_0 \left( h(s) f \left( u(\sigma(s)) \right) \right) \Delta s \\
\leq B_2 \varphi_0 \left( h(b) f (m) \right) + \sum_{\Delta} \varphi_0 \left( h(s) f (m) \right) \Delta s \\
\leq dm B_2 \varphi_0 (h(b)) + dm \sum_{\Delta} \varphi_0 (h(s)) \Delta s \leq m = \|u\|.
\]

Thus it is shown that in either instance, if we take \( S_m = \{ u \in E : \|u\| < m \} \), then we have \( \|Lu\| \leq \|u\| \) for \( u \in P \cap \partial S_m \). It follows from Theorem 3.1 (ii) that \( L \) has a fixed point \( u \) in \( P \cap (\overline{S_m} \setminus S_k) \) with \( k \leq \|u\| \leq m \). □

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References