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Existence of solutions for first-order multi-point problems with changing-sign nonlinearity

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In this study, conditions for the existence of at least one positive solution to a nonlinear first-order multi-point eigenvalue problem on time scales are discussed. Here the nonlinearity is allowed to take on negative values. The results, which are new for differential/difference equations as well as arbitrary time scales, are based on the Guo–Krasnosel’skiĭ fixed point theorem.

Keywords: time scales; positive solutions; cone; fixed point theorem

2000 Mathematics Subject Classification: 34B05; 39A10

1. Introduction

We are interested in the first-order $n$-point time-scale boundary value problem

$$
y^\Delta(t) + p(t)y^\sigma(t) = \lambda f(t, y^\sigma(t)), \quad t \in (a, b),
$$

$$
y(a) = y(b) + \sum_{i=2}^{n-1} \gamma_i y(t_i),
$$

where $n \geq 3$ and

$$
p : [a, b] \rightarrow (0, \infty), \quad p \in C_0[a, b];
$$

the points $t_i \in \mathbb{T}$ for $i \in \{1, 2, \ldots, n\}$ with $a < t_2 < \cdots < t_{n-1} < b \in \mathbb{T}$; the real scalar $\lambda \in (0, \infty)$; $\gamma_i \in [0, \infty)$ for $i \in \{2, \ldots, n-1\}$;

$$
d := e_p(b, a) - 1 > 0, \quad d > \sum_{i=2}^{n-1} \gamma_i e_p(b, t_i) \geq 0;
$$

the continuous function $f : (a, b) \times [0, \infty) \rightarrow (-\infty, \infty)$ is such that the following hold:

$$
\lim_{y \rightarrow +\infty} f(t, y) = +\infty, \quad t \in [t_2, t_3]; \quad -u(t) \leq f(t, y) \leq z(t) h(y)
$$

for right-dense continuous functions $u, z : (a, b) \rightarrow (0, \infty)$ and continuous function $h : [0, \infty) \rightarrow (0, \infty)$.

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Problem (1.1), (1.2) is new to all time scales, to the best of our knowledge including the discrete case $\mathbb{T} = \mathbb{Z}$ and the continuous case $\mathbb{T} = \mathbb{R}$. Some of the motivation for this paper comes from second-order problems in Zhang et al [16], which extends the discussion found in Ma and Thompson [10]. See also related second-order and third-order time-scale boundary value problems found in [1,3]. Moreover, there has of late been interest in first-order problems on time scales. Cabada and Vivero [6], Dai and Tisdell [7], Otero-Espinar and Vivero [11], Sun [12], Sun and Li [13,14], and Tian and Ge [15] all recently consider first-order boundary value problems on time scales, but none of them consider the multi-point problem or the possibility of a changing-sign nonlinearity. Throughout this paper we assume a knowledge of time scales and time-scale notation; for more general information concerning dynamic equations on time scales, introduced by Aulbach and Hilger [4] and Hilger [9], see the excellent text by Bohner and Peterson [5].

2. Linear preliminaries

We initially construct the Green function for the linear first-order periodic boundary value problem

$$y^A(t) + p(t)y^q(t) = \lambda u(t), \quad a < t < b, \quad (2.1)$$

$$y(a) = y(b) \quad (2.2)$$

assuming (1.3) and (1.4), and where $u$ is as in (1.5). The techniques here are standard when establishing the Green function for a boundary value problem.

**Lemma 2.1.** Assume (1.3)–(1.5). Then the nonhomogeneous boundary value problem (2.1), (2.2) has a unique solution $y$ for which the formula

$$y(t) = \lambda \int_a^b G(t, s)u(s)\Delta s, \quad t \in [a, b]_\mathbb{T}$$

holds, where the function $G(t, s)$ is given by

$$G(t, s) = \begin{cases} \frac{1}{d} e_p(s, t) & : a \leq t \leq s \leq b, \\ \frac{d}{d} + 1) e_p(s, t) & : a \leq s < t \leq b, \end{cases} \quad (2.3)$$

and $G(t, s)$ is the Green function of the boundary value problem (2.1), (2.2).

**Proof.** Set

$$y(t) = \lambda \int_a^b G(t, s)u(s)\Delta s = \lambda \left( \frac{1}{d} + 1 \right) \int_a^t e_p(s, t)u(s)\Delta s + \frac{\lambda}{d} \int_t^b e_p(s, t)u(s)\Delta s.$$ 

Then

$$y(a) = \frac{\lambda}{d} \int_a^b e_p(s, a)u(s)\Delta s \quad \text{and} \quad y(b) = \lambda \left( \frac{1}{d} + 1 \right) \int_a^b e_p(b, s)u(s)\Delta s.$$
so that
\[ y(b) - y(a) = \frac{\lambda}{d} \int_a^b [e_p(s, b) + de_p(s, b) - e_p(s, a)]u(s)\Delta s \]
\[ = \frac{\lambda}{d} \int_a^b [e_p(s, b)e_p(b, a) - e_p(s, a)]u(s)\Delta s = 0 \]
by the semi-group property of the time-scale exponential function ([5], Theorem 2.36 (v)) using \( d \) given in (1.4). Thus, the boundary condition is satisfied. Taking a delta derivative and recalling that \( e_p(s, t) = e_{cp}(t, s) \) by ([5], Theorem 2.36 (iv)),
\[ y^\Delta(t) = \lambda e_p(t, \sigma(t))u(t) + \Theta p(t)\lambda \int_a^t e_p(s, t)u(s)\Delta s + \frac{\Theta p(t)\lambda}{d} \int_a^b e_p(s, t)u(s)\Delta s. \]
We also have that \( e_p(s, \sigma(t)) = e_p(s, t)/(1 + \mu(t)p(t)) \), so that
\[ p(t)y^\sigma(t) = \frac{-\Theta p(t)\lambda}{d} \int_a^b e_p(s, t)u(s)\Delta s - \Theta p(t)\lambda \left[ \int_a^t e_p(s, t)u(s)\Delta s + \mu(t)u(t) \right]. \]
It follows that
\[ y^\Delta(t) + p(t)y^\sigma(t) = \frac{\lambda u(t)}{1 + \mu(t)p(t)} - \Theta p(t)\lambda \mu(t)u(t) = \lambda u(t), \]
and the result is proven. \( \square \)

**Lemma 2.2.** Let (1.3) and (1.4) hold. Then the Green function \( G(t, s) \) in (2.3) satisfies
\[ 0 < e_p(s, b)G(s, s) \leq G(t, s) \leq e_p(s, a)G(s, s) \] (2.4)
for any \( s, t \in [a, b]_T \).

**Proof.** Fix \( s \in [a, b]_T \). For \( t \leq s \), the function \( e_p(s, t) \) is decreasing in \( t \), ensuring that
\[ \frac{1}{d}e_p(s, s) \leq \frac{1}{d}e_p(s, t) \leq \frac{1}{d}e_p(s, a), \]
that is
\[ G(s, s) \leq G(t, s) \leq e_p(s, a)G(s, s). \]
For \( t > s \),
\[ \left( \frac{1}{d} + 1 \right)e_p(s, b) \leq \left( \frac{1}{d} + 1 \right)e_p(s, t) \leq \left( \frac{1}{d} + 1 \right)e_p(s, s), \]
rewritten as
\[ e_p(s, b)G(s, s) \leq G(t, s) \leq G(s, s). \]
Thus for any \( t, s \in [a, b] \),
\[
\min\{1, e_p(s, b)\} G(s, s) \leq G(t, s) \leq \max\{1, e_p(s, a)\} G(s, s).
\]

Since \( p > 0 \), the result follows. \( \square \)

**Lemma 2.3.** Assume \((1.3)-(1.5)\). If \( \int_a^b G(s, s)u(s)\Delta s < \infty \), then the nonhomogeneous dynamic equation \((2.1)\) with boundary conditions \((1.2)\) has a unique solution \( w \) for which the formula
\[
w(t) = \lambda \left( \int_a^b G(t, s)u(s)\Delta s + A(u)e_p(a, t) \right), \quad t \in [a, b]
\]
holds, where the function \( G(t, s) \) is the Green function \((2.3)\) of the boundary value problem \((2.1), (2.2)\) and the functional \( A \) is defined by
\[
A(u) := \frac{e_p(b, a)}{d - \sum_{i=2}^{n-1} \gamma_i e_p(b, ti)} \sum_{i=2}^{n-1} \gamma_i \int_a^b G(ti, s)u(s)\Delta s.
\]

**Proof.** It can be verified that for a solution \( w \) of the nonhomogeneous equation \((2.1)\) under the nonhomogeneous boundary conditions \((1.2)\), the formula \((2.5)\) holds, where \( G(t, s) \) is given by \((2.3)\). We thus show that the function \( w \) given in \((2.5)\) is a solution of \((2.1)\) with condition \((1.2)\) only if \( A \) is given by \((2.6)\). If \( w \) as in \((2.5)\) is a solution of \((2.1), (1.2)\), then
\[
w(t) = \lambda \left( \int_a^b G(t, s)u(s)\Delta s + Ae_p(a, t) \right)
\]
for some constant \( A \). Then
\[
w(a) - w(b) = \lambda A(1 - e_p(a, b)),
\]
and
\[
\sum_{i=2}^{n-1} \gamma_i w(ti) = \lambda \sum_{i=2}^{n-1} \gamma_i \left( \int_a^b G(ti, s)u(s)\Delta s + Ae_p(a, ti) \right).
\]
To satisfy condition \((1.2)\),
\[
A(1 - e_p(a, b)) = \sum_{i=2}^{n-1} \gamma_i \left( \int_a^b G(ti, s)u(s)\Delta s + Ae_p(a, ti) \right);
\]
multiply both sides by \( e_p(b, a) \) and solve for \( A \) to get \((2.6)\). \( \square \)

**Remark 2.4.** Note that \( A(u) \) given in \((2.6)\) satisfies \( A(u) > 0 \) for \( u > 0 \) by \((1.4)\) and \((2.4)\).
Lemma 2.5. Let (1.3)–(1.5) hold. Then the unique solution \( w \) as in (2.5) of the problem (2.1), (1.2) satisfies

\[
e_p(a,b)\|w\| \leq w(t) \leq \lambda e_p(a,b)\xi, \quad t \in [a,b], \quad \|w\| = \max_{t \in [a,b]} w(t),
\]

where

\[
\xi := e_p(b,a) \left( 1 + \frac{e_p(b,t)\sum_{i=2}^{n-1} \gamma_i}{d - \sum_{i=2}^{n-1} \gamma_i e_p(b,t_i)} \right) \int_a^b e_p(s,a)G(s,s)u(s)\Delta s.
\]

Proof. From Lemma 2.2, the Green function (2.3) satisfies \( 0 < G(t,s) \leq e_p(s,a)G(s,s) \) for \( t \in [a,b] \), so that for all \( t,t_i \in [a,b] \),

\[
w(t) \leq \lambda \left( \int_a^b e_p(s,a)G(s,s)u(s)\Delta s + \frac{e_p(b,t)}{d - \sum_{i=2}^{n-1} \gamma_i e_p(b,t_i)} \int_a^b e_p(s,a)G(s,s)u(s)\Delta s \right)
\]

\[
\leq \lambda \left( 1 + \frac{e_p(b,t)\sum_{i=2}^{n-1} \gamma_i}{d - \sum_{i=2}^{n-1} \gamma_i e_p(b,t_i)} \right) \int_a^b e_p(s,a)G(s,s)u(s)\Delta s = \lambda e_p(a,b)\xi,
\]

for \( \xi \) as in (2.7). For all \( t \in [a,b] \),

\[
w(t) \geq \lambda \left( \int_a^b e_p(s,a)e_p(a,b)G(s,s)u(s)\Delta s + A(u)e_p(a,b)e_p(b,t) \right)
\]

\[
\geq e_p(a,b)\lambda \left( \int_a^b e_p(s,a)G(s,s)u(s)\Delta s + A(u)e_p(b,t) \right) \geq e_p(a,b)\|w\|.
\]

This completes the proof of the lemma. \( \square \)

3. Existence result

Let \( B \) denote the Banach space \( C_{ad}[a,b] \) with the norm \( \|y\| = \sup_{t \in [a,b]} |y(t)| \). Define the cone \( \mathcal{P} \subset B \) by

\[
\mathcal{P} = \{ y \in B : y(t) \geq e_p(a,b)\|y\| \text{ on } [a,b] \}.
\]

The following is a generalization of the discussion by Zhang et al [16] to arbitrary time scales; see also Ref. [1]. Consider the related boundary value problem

\[
y^{(d)}(t) + p(t)y^{(n)}(t) = f_w(t,y^{(n)}(t)), \quad t \in (a,b), \quad y(a) = \sum_{i=2}^{n-1} \gamma_i y(t_i) + y(b),
\]

where

\[
f_w(t,y^{(n)}(t)) := f(t,y^{(n)}_w(t)) + u(t), \quad y_w(t) := \max\{y(t) - w(t), 0\}
\]

such that \( w \) given in (2.5) is the solution of (2.1), (1.2).
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For any fixed \( y \in \mathcal{P}, \ y_w \equiv y \leq \|y\| \) and by (1.5),

\[
\int_a^b G(t, s)f_w(s, y^\sigma(s))\Delta s \leq \int_a^b e_p(s, a)G(s, s)(z(s)h(y^\sigma_w(s)) + u(s))\Delta s
\]

\[
\leq (\max_{0\leq \tau \leq \|y\|} h(\tau) + 1) \int_a^b e_p(s, a)G(s, s)(z(s) + u(s))\Delta s.
\]

Additionally, using the properties of the Green function (2.4), for \( i = 2, \ldots, n - 1 \) we have

\[
\int_a^b G(t, s)(z(s) + u(s))\Delta s \leq \int_a^b e_p(s, a)G(s, s)(z(s) + u(s))\Delta s.
\]

Thus it follows that, for \( A \) as in (2.6),

\[
A(z + u) = \frac{e_p(b, a)}{d - \sum_{i=2}^{n-1} \gamma_i e_p(b, t_i)} \sum_{i=2}^{n-1} \gamma_i \int_a^b G(t, s)(z(s) + u(s))\Delta s
\]

\[
\leq \frac{e_p(b, a)\sum_{i=2}^{n-1} \gamma_i}{d - \sum_{i=2}^{n-1} \gamma_i e_p(b, t_i)} \int_a^b e_p(s, a)G(s, s)(z(s) + u(s))\Delta s < \infty,
\]

if in the rest of the discussion we make the additional assumption that

\[
\int_a^b e_p(s, a)G(s, s)(z(s) + u(s))\Delta s < \infty. \tag{3.2}
\]

This allows us to define for \( y \in \mathcal{P} \) the operator \( T: \mathcal{P} \to \mathcal{B} \) given via

\[
(Ty)(t) := \lambda \left( \int_a^b G(t, s)f_w(s, y^\sigma(s))\Delta s + A(f_w)e_p(a, t) \right) \tag{3.3}
\]

using (2.6) and (3.1).

**Lemma 3.1.** Assume that (1.3)–(1.5) and (3.2) hold. Then \( T: \mathcal{P} \to \mathcal{P} \) is completely continuous.

**Proof.** For any \( y \in \mathcal{P}, \) (2.4) implies that for all \( t \in [a, b]_\mathbb{T}, \)

\[
(Ty)(t) \leq \lambda \left( \int_a^b e_p(s, a)G(s, s)f_w(s, y^\sigma(s))\Delta s + A(f_w)e_p(a, t) \right).
\]

On the other hand, using another inequality from (2.4),

\[
(Ty)(t) \geq e_p(a, b)\lambda \left( \int_a^b e_p(s, a)G(s, s)f_w(s, y^\sigma(s))\Delta s + A(f_w)e_p(b, t) \right).
\]

Therefore, \( (Ty)(t) \geq e_p(a, b)\|Ty\| \) on \( [a, b]_\mathbb{T}, \) ensuring that \( T(\mathcal{P}) \subseteq \mathcal{P}. \) By a standard application of the Arzela–Ascoli Theorem, \( T \) is completely continuous. \( \square \)
To establish an existence result we will employ the following fixed point theorem due to Krasnosel’ skii [8], and seek a fixed point of $T$ in $\mathcal{P}$.

**Theorem 3.2.** Let $E$ be a Banach space, $P \subseteq E$ be a cone, and suppose that $S_1, S_2$ are bounded open balls of $E$ centered at the origin with $\bar{S}_1 \subseteq S_2$. Suppose further that $L : P \cap (\bar{S}_2 \setminus S_1) \rightarrow P$ is a completely continuous operator such that either

(i) $\|Ly\| \leq \|y\|$, $y \in P \cap \partial S_1$ and $\|Ly\| \geq \|y\|$, $y \in P \cap \partial S_2$, or

(ii) $\|Ly\| \geq \|y\|$, $y \in P \cap \partial S_1$ and $\|Ly\| \leq \|y\|$, $y \in P \cap \partial S_2$

holds. Then $L$ has a fixed point in $P \cap (\bar{S}_2 \setminus S_1)$.

**Theorem 3.3.** Assume that (1.3)–(1.5) and (3.2) hold. Then there exists $\lambda^* > 0$ such that the first-order multi-point time scale boundary value problem (1.1), (1.2) has at least one positive solution in $\mathcal{P}$ for any $\lambda \in (0, \lambda^*)$.

**Proof.** By Lemma 3.1, $T : \mathcal{P} \rightarrow \mathcal{P}$ given by (3.3) is completely continuous. Take $S_1 := \{y \in B : \|y\| < \xi\}$, for $\xi$ given (2.7). Let

$$
\lambda^* := \min \left\{ \frac{\int_{a}^{b} e_p(s,a)G(s,s)u(s) \Delta s}{\max_{0 \leq \tau \leq \xi} h(\tau) + 1} \right\}.
$$

Then for any $y \in \mathcal{P} \cap \partial S_1$, $0 \leq y_u(s) \leq y(s) \leq \|y\| = \xi$ for $s \in [a,b]_\mathbb{T}$, and

$$(Ty)(t) \leq \lambda \int_{a}^{b} e_p(s,a) G(s,s) \left( z(s) h(y_u^+(s)) + u(s) \right) \Delta s
\leq \lambda \left( 1 + \frac{e_p(b,t) \sum_{i=2}^{n-1} \gamma_i}{d - \sum_{i=2}^{n-1} \gamma_i e_p(b,t)} \right) \left( \max_{0 \leq \tau \leq \xi} h(\tau) + 1 \right)
\times \int_{a}^{b} e_p(s,a) e_p(a,b) G(s,s)(z(s) + u(s)) \Delta s \leq \xi = \|y\|.
$$

Hence $\|Ty\| \leq \|y\|$ for $y \in \mathcal{P} \cap \partial S_1$. Pick $K \in \mathbb{R}$ such that $K > 0$ and

$$
1 \leq \frac{\lambda K e_p(a,b)}{\xi + 1} \min_{t_1 \leq \tau \leq t_2} \int_{t_1}^{t_2} G(t,s) \Delta s.
$$

By (1.5), for any $t \in [t_2, t_3]_\mathbb{T}$, there exists a constant $N > 0$ such that $f(t,y) > Ky$ for $y > N$. Pick $Q := \max \left\{ \lambda(\xi + 1), \xi + 1, \frac{N(\xi + 1)}{\gamma(a,b)} \right\}$. If $S_2 := \{y \in B : \|y\| < Q\}$, then for any
so that

\[ y'(t) - w'(t) \geq y'(t) - \lambda e_p(a, b) \xi \geq y'(t) - \frac{\lambda \xi}{Q} y'(t) \geq \left(1 - \frac{\lambda \xi}{Q}\right)y'(t) \]

\[ \geq \left(1 - \frac{\lambda \xi}{\lambda + 1}\right)y'(t) = \frac{y'(t)}{\xi + 1} \geq 0. \]

Thus

\[ \min_{t \in [t_1, t_2]} (y(t) - w(t)) \geq \min_{t \in [t_1, t_2]} \frac{y(t)}{\xi + 1} = \frac{e_p(a, b)Q}{\xi + 1} \geq N, \]

so that

\[
\begin{align*}
\min_{t \in [t_1, t_2]} (Ty)(t) &= \min_{t \in [t_1, t_2]} \lambda \left( \int_a^b G(t, s) f_w(s, y'(s)) \Delta s + A(f_w) e_p(a, t) \right) \\
&\geq \lambda \min_{t \in [t_1, t_2]} \int_{t_2}^{t_1} G(t, s) f_w(s, y'(s)) \Delta s \\
&\geq \lambda K \min_{t \in [t_1, t_2]} \int_{t_2}^{t_1} G(t, s)(y'(s) - w'(s)) \Delta s \\
&\geq \frac{\lambda Ke_p(a, b)Q}{\xi + 1} \min_{t \in [t_1, t_2]} \int_{t_2}^{t_1} G(t, s) \Delta s \\
&= \frac{\lambda Ke_p(a, b)\|y\|}{\xi + 1} \min_{t \in [t_1, t_2]} \int_{t_2}^{t_1} G(t, s) \Delta s \geq \|y\|. 
\end{align*}
\]

Hence for \( y \in \mathcal{P} \cap \bar{a}S_2 \) we have \( \|Ty\| \geq \|y\| \). By Theorem 3.2, \( T \) has a fixed point \( y \) such that \( \xi \leq \|y\| \leq Q \). But then

\[ y'(t) - w'(t) \geq e_p(a, b) \xi - \lambda e_p(a, b) \xi \geq (1 - \lambda)e_p(a, b) \xi \geq 0. \]

As a consequence, this \( y \) solves the boundary value problem

\[ y^\Delta(t) + p(t)y^\sigma(t) = \lambda \left( f(t, y^\sigma(t)) + u(t) \right), \quad t \in (a, b) \setminus \{a, b\}, \]

\[ y(a) = \sum_{i=2}^{n-1} \gamma_i y(t_i) + y(b). \]

Now set \( x(t) := y(t) - w(t) \) for \( w \) given in (2.5). Then \( y^\Delta = x^\Delta + w^\Delta \). As \( w \) is the solution of (2.1), (2.2), we see that

\[ x^\Delta(t) + p(t)x^\sigma(t) = \lambda f(t, x^\sigma(t)), \quad t \in (a, b) \setminus \{a, b\}, \quad x(a) = \sum_{i=2}^{n-1} \gamma_i x(t_i) + x(b), \]

in other words, \( x \) is a positive solution of the first-order multi-point time scale boundary value problem (1.1), (1.2). \( \square \)
Remark 3.4. The boundary condition in (1.2) could be changed to the nonlocal condition

\[ y(a) = y(b) + \int_{\tau_1}^{\tau_2} \gamma(t)y(t) \Delta t, \]

for \( \gamma \in C_{ad}([a, b]_T, [0, \infty)) \) and \( \tau_1, \tau_2 \in [a, b]_T \) with \( \tau_1 < \tau_2 \). Condition (1.2) could also be slightly generalized to

\[ \alpha y(a) = \beta y(b) + \sum_{i=2}^{n-1} \gamma_i y(t_i), \]

where \( \alpha, \beta \in \mathbb{R} \) such that \( 0 < \frac{\beta}{\alpha \rho(b,a)} < 1 \); in this latter case (2.3) becomes

\[
G(t, s) = \begin{cases} 
\frac{\beta e_{p}(s,t)}{\alpha e_{p}(b,a) - \beta} & : a \leq t \leq s \leq b, \\
\frac{\alpha e_{p}(b,a) e_{p}(s,t)}{\alpha e_{p}(b,a) - \beta} & : a \leq s < t \leq b.
\end{cases}
\]

For either change in (1.2) the analysis would be similar to that given in this paper.

Remark 3.5. Using Lemmas 2.1, 2.2 and 2.5, respectively, we can easily extend the existence and uniqueness results found in Dai and Tisdell [7], Sun [12], Sun and Li [13,14], and Tian and Ge [15] to the multi-point problem (1.1), (1.2).

Remark 3.6. Base boundary conditions involving the delta derivative which can then be extended to multi-point problems are also possible. For example, if

\[ y^\Delta (a) = \beta y^\Delta (b), \]

then

\[
G(t, s) = \begin{cases} 
k e_{p}(s,t) & : a \leq t \leq s \leq b, \\
(k+1) e_{p}(s,t) & : a \leq s < t \leq b
\end{cases}
\]

for

\[ 0 < \beta < \frac{p(a) e_{p}(\sigma(b), \sigma(a))}{p(b)} \quad \text{and} \quad k = \frac{-\beta p(b)}{\beta p(b) + \sum p(a) e_{p}(\sigma(b), a)} > 0. \]

Remark 3.7. Similar results can be obtained for the related first-order delta dynamic equations

\[
y^\Delta (t) + p(t)y(t) = \lambda f(t, y(t)), \quad t \in [a, b]_T, \\
y^\Delta (t) + (\sum p(t)) \sigma_y(t) = -\lambda f(t, y^\sigma(t)), \quad t \in [a, b]_T, \\
y^\Delta (t) - \sum (-p(t))y(t) = -\lambda f(t, y(t)), \quad t \in [a, b]_T,
\]
and for the corresponding first-order nabla dynamic equations

\begin{align*}
y^\nabla(t) + p(t)y^\rho(t) &= \lambda f(t, y^\rho(t)), \quad t \in [a, b], \\
y^\nabla(t) + p(t)y(t) &= \lambda f(t, y(t)), \quad t \in [a, b], \\
y^\nabla(t) + (\nabla^\rho p(t))y^\rho(t) &= -\lambda f(t, y^\rho(t)), \quad t \in [a, b], \\
y^\nabla(t) - \nabla_{\Delta}(-p)(t)y(t) &= -\lambda f(t, y(t)), \quad t \in [a, b],
\end{align*}

see [2].

References