Twin monotone positive solutions to a singular nonlinear third-order differential equation

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Abstract

This paper discusses the existence of at least one or two nondecreasing positive solutions for the following singular nonlinear third-order differential equation

\[ x'''(t) + \lambda \alpha(t)f(t, x(t)) = 0, \quad a < t < b, \]
\[ x(a) = x''(a) = x'(b) = 0, \]

where \( \lambda > 0 \) is a parameter, \( \alpha \in C((a, b), \mathbb{R}^+) \), \( f \in C([a, b] \times (0, +\infty), \mathbb{R}^+) \), \( \alpha(t) \) may be singular at \( t = a, b \) and \( f(t, s) \) may be singular at \( s = 0 \). Green’s function and the fixed-point theorem of cone expansion and compression type are utilized. To illustrate the results, four nontrivial examples are included. © 2006 Elsevier Inc. All rights reserved.

Keywords: Singular nonlinear third-order differential equation; Positive solutions; Green’s function; Fixed-point theorem of cone expansion and compression type

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1. Introduction

In this paper we are concerned with the existence of positive solutions for the following singular nonlinear third-order ordinary differential equation

\[ x'''(t) + \lambda \alpha(t) f(t, x(t)) = 0, \quad a < t < b, \]  
\[ x(a) = x''(a) = x'(b) = 0 \]  

(1.1)

under the conditions that \( \lambda > 0 \) is a parameter, \( \alpha \in C((a, b), \mathbb{R}^+) \), \( f \in C([a, b] \times (0, +\infty), \mathbb{R}^+) \) and

\[ \alpha(t) \text{ may be singular at } t = a, b \quad \text{and} \quad f(t, s) \text{ may be singular at } s = 0. \]  

(1.3)

In the past ten years or so, various boundary value problems for ordinary differential equations have been studied extensively; see, for example, [1–11,13–22] and the references therein. Dang and Schmitt [6] and Erbe, Hu and Wang [8] gave several sufficient conditions for the existence of solutions for the nonlinear second-order ordinary differential equation

\[ x''(t) = f(t, x(t)), \quad 0 < t < 1, \]  

(1.4)

under boundary conditions \( x(0) = x(1) = 0 \). Using the Krasnosel’skii fixed-point theorem, Henderson and Wang [13] studied the existence of positive solutions for the nonlinear eigen-value problem (1.4) under boundary conditions \( x(0) = x(1) = 0 \) and the assumptions that \( f \in C([0, 1], \mathbb{R}^+) \) and \( f(0) = \lim_{t \to 0^+} f(t) \) and \( f(\infty) = \lim_{t \to \infty} f(t) \) exist. Utilizing the Leray–Schauder degree theory and lower and upper solutions method, Du, Ge and Liu [7] investigated the existence of solutions for the third-order nonlinear boundary value problem

\[ x'''(t) = f(t, x(t), x'(t), x''(t)), \quad 0 < t < 1, \]  

(1.5)

under two-point nonlinear boundary conditions

\[ x(0) = 0, \quad g(x'(0), x''(0)) = A, \quad h(x'(1), x''(1)) = B, \]  

where \( A, B \in \mathbb{R} \), \( f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R} \) is continuous, \( g, h : \mathbb{R}^2 \to \mathbb{R} \) are continuous. Depending on degree theory, and lower and upper solutions method, Grossinho, Minhós and Santos [10,11] studied the existence and location results for the third-order separated boundary value problem

\[ x'''(t) = f(t, x(t), x'(t), x''(t)), \quad a \leq t \leq b, \]  

(1.6)

under four types of two-point nonlinear boundary conditions

\[ x(a) = A, \quad x''(a) = B, \quad x''(b) = C, \quad \text{or} \]  
\[ x(a) = A, \quad C_1 x'(a) - C_2 x''(a) = B, \quad C_3 x'(b) + C_4 x''(b) = C, \quad \text{or} \]  
\[ x(a) = A, \quad \varphi(x'(b), x''(b)) = 0, \quad x''(a) = B, \quad \text{or} \]  
\[ x(a) = A, \quad \psi(x'(a), x''(a)) = 0, \quad x''(b) = C, \]  

(1.7)

(1.8)

(1.9)

(1.10)

where \( C_1, C_2, C_3, C_4 > 0 \), \( A, B, C \in \mathbb{R} \), \( f : [a, b] \times \mathbb{R}^3 \to \mathbb{R} \) is a continuous function, \( \varphi, \psi : \mathbb{R}^2 \to \mathbb{R} \) are continuous functions, monotone in the second variable. Cabada [5] and Yao and Feng [22] used the method of lower and upper solutions and fixed-point theorems to establish the existence of periodic solutions and positive solutions for the third-order ordinary differential equation

\[ x'''(t) - f(t, x(t)) = 0, \quad a \leq t \leq b, \]  

(1.11)
under two-point boundary conditions $u^{(i)}(0) = u^{(i)}(2\pi)$, $i = 0, 1, 2$, with $a = 0$, $b = 2\pi$, and $x(0) = x'(0) = x''(1) = 0$ with $a = 0$, $b = 1$, respectively. Using the Krasnosel’kii, Leggett–Williams and five functionals fixed-point theorems, Anderson [2], Anderson and Davis [3], Wong [20], and Yao [21] gave some sufficient conditions for the existence of multiple positive solutions to Eq. (1.11) under three-point boundary conditions $x(t_1) = x'(t_2) = 0$, $\gamma x(t_3) + \delta x''(t_3) = 0$, or $x(t_1) = x'(t_2) = x''(t_3) = 0$ with $a = t_1$ and $b = t_3$, or $x(0) = x'(\eta) = x''(1) = 0$ with $a = 0$ and $b = 1$, respectively. Very recently, applying the Krasnosel’kii fixed-point theorem, Sun [17] and Li [16] obtained the existence of multiple positive solutions for the singular nonlinear third-order differential equations

$$x'''(t) - \lambda \alpha(t)f(t, x(t)) = 0, \quad 0 < t < 1,$$

(1.12)

under three-point boundary conditions $x(0) = x'(\eta) = x''(1) = 0$, and

$$x'''(t) - \lambda \alpha(t)f(x(t)) = 0, \quad 0 < t < 1,$$

(1.13)

under two-point boundary conditions $x(0) = x'(0) = x''(1) = 0$, respectively, where the function $\alpha(t)$ may be singular at $t = 0, 1$ and the function $f$ has no singularity.

However, to the authors’ knowledge, few papers in the literature can be found dealing with the existence of positive solutions to the boundary value problem (1.1), (1.2) under the singularity assumptions (1.3). The purpose of this paper is to fill in the gap in this area. By means of the positivity of Green’s function $G(t, s)$ and the fixed-point theorem of cone expansion and compression type, we establish a few sufficient conditions for the existence of at least one or two nondecreasing positive solutions to the boundary value problem (1.1), (1.2) under the singularity assumptions (1.3) and certain conditions.

Although (1.1), (1.2) are particular cases of (1.6), (1.9), we cannot invoke the results in [10] to show the existence of solutions for the boundary value problem (1.1), (1.2) under the singularity assumptions (1.3). On the other hand, using similar arguments and techniques, the results presented in this paper could be extended to $n$th-order boundary value problems and other types of two-point boundary conditions. Of course, we shall continue to study these possible extensions and generalizations in the future.

The results in this paper are organized as follows. Section 2 introduces some notation and several lemmas which play important roles in this paper. Section 3 provides a few sufficient conditions for the existence of at least one or two nondecreasing positive solutions for boundary value problem (1.1), (1.2). Applications of the existence results to four nontrivial examples are presented in Section 4.

2. Preliminaries and lemmas

Let $X$ be a real Banach space and $Y$ be a cone in $X$. Set

$$Y_r = \left\{ x \in Y : \|x\| < r \right\}, \quad \partial Y_r = \left\{ x \in Y : \|x\| = r \right\}, \quad r > 0,$$

$$Y_{r,s} = \left\{ x \in Y : r \leq \|x\| \leq s \right\}, \quad s > r > 0.$$ 

A function $x$ is said to be a solution of the boundary value problem (1.1), (1.2) if $x \in C^2([a, b], \mathbb{R}) \cap C^3((a, b), \mathbb{R})$ satisfies Eq. (1.1) and boundary conditions (1.2). A function $x$ is said to be a positive solution of (1.1), (1.2) if $x$ is a solution of Eq. (1.1) and boundary conditions (1.2) with $x(t) > 0$ for each $t \in (a, b)$. Throughout this paper, we assume that $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ are strictly decreasing and strictly increasing sequences, respectively, with $\lim_{n \to \infty} a_n = a$, $\lim_{n \to \infty} b_n = b$ and $a_1 < b_1$. 


\[ D_n = \min \left\{ \frac{a_n - a}{2(b - a)}, \frac{b - b_n}{2(b - a)} \right\}, \quad A_n = [a, a_n] \cup [b_n, b], \quad n \geq 1, \]

\[ h(t) = \frac{t - a}{2(b - a)}, \quad g(t) = (b - a)(b - t), \quad t \in [a, b], \]

\[ P = \left\{ x \in C([a, b], \mathbb{R}) : x(t) \geq h(t)\|x\|, \quad t \in [a, b] \right\}, \quad (2.1) \]

where \( \|x\| = \sup_{t \in [a, b]} |x(t)| \) for each \( x \in C([a, b], \mathbb{R}) \), and

\[ G(t, s) = \begin{cases} 
(t - a)(b - s) - \frac{1}{2}(t - s)^2, & a \leq s \leq t \leq b, \\
(t - a)(b - s), & a \leq t < s \leq b,
\end{cases} \]

is Green’s function for the homogeneous problem \( x'''(t) = 0 \) satisfying boundary conditions (1.2). It is easy to verify that \( P \) is a cone of the Banach space \( C([a, b], \mathbb{R}) \). Moreover, we use also the following notation and assumptions:

\[ k = \left[ \int_a^b g(t)\alpha(t) \, dt \right]^{-1}, \quad m = \left[ h(p) \int_p^q g(t)\alpha(t) \, dt \right]^{-1}, \]

\[ M(s) = \sup_{x \in \partial P_s} \lambda \int_a^b g(t)\alpha(t)f(t, x(t)) \, dt, \quad s > 0, \]

\[ L(\beta, r) = \max \left\{ \frac{\beta}{h(p)}, \frac{kM(\beta)}{k - r} \right\}, \quad 0 < \beta, r < k, \]

\[ f_0 = \lim_{s \to 0^+} \min_{t \in [p, q]} \frac{f(t, s)}{s}, \quad f_0 = \liminf_{s \to 0^+} \min_{t \in [p, q]} \frac{f(t, s)}{s}, \]

\[ f_\infty = \lim_{s \to +\infty} \min_{t \in [p, q]} \frac{f(t, s)}{s}, \quad f_\infty = \liminf_{s \to +\infty} \min_{t \in [p, q]} \frac{f(t, s)}{s}, \]

\[ f^\infty = \lim_{s \to +\infty} \max_{t \in [a, b]} \frac{f(t, s)}{s}, \quad f^\infty = \limsup_{s \to +\infty} \max_{t \in [a, b]} \frac{f(t, s)}{s}, \]

(C1) \( \alpha : (a, b) \to \mathbb{R}^+ \) is continuous and

\[ \int_p^q g(t)\alpha(t) \, dt > 0 \quad \text{and} \quad \int_a^b g(t)\alpha(t) \, dt < +\infty, \]

where \( p \) and \( q \) are constants with \( a < p < q \leq b \), and

(C2) \( f : [a, b] \times (0, +\infty) \to \mathbb{R}^+ \) is continuous and

\[ \lim_{n \to +\infty} \sup_{x \in P_{c,d}} \int_{A_n} g(t)\alpha(t)f(t, x(t)) \, dt = 0 \quad \text{for all } c, d \text{ with } 0 < c < d. \]

**Lemma 2.1.** Green’s function satisfies \( 0 \leq h(t)g(s) \leq G(t, s) \leq g(s), \quad t, s \in [a, b] \).

**Proof.** Let \( t, s \) be in \([a, b]\). For \( s < t \), we get that
\[ 0 \leq h(t)g(s) = (t-a)(b-s) - \frac{1}{2}(t-a)(b-s) \leq G(t,s) = (t-a)(b-s) - \frac{1}{2}(t-s)^2 \leq g(s). \]

For \( t < s \), we have
\[ 0 \leq h(t)g(s) = \frac{1}{2}G(t,s) \leq g(s). \]

This completes the proof. \( \square \)

**Lemma 2.2.** Let \((C_1)\) and \((C_2)\) hold and \(c, d\) be fixed constants with \(0 < c < d\). Define an integral operator \(T_\lambda : P_{c,d} \rightarrow P\) by
\[ T_\lambda x(t) = \lambda \int_a^b G(t,s)\alpha(s)f(s,x(s))\,ds, \quad (x,t) \in P_{c,d} \times [a,b]. \tag{2.2} \]

Then \(T_\lambda\) is completely continuous.

**Proof.** First of all we claim that
\[ \sup_{x \in P_{c,d}} \lambda \int_a^b g(t)\alpha(t)f(t,x(t))\,dt < +\infty. \tag{2.3} \]

It follows from \((C_2)\) that there exists some positive integer \(n_0\) satisfying
\[ \sup_{x \in P_{c,d}} \lambda \int_{A_{n_0}} g(t)\alpha(t)f(t,x(t))\,dt < 1. \tag{2.4} \]

It is easy to see that for each \(x \in P_{c,d}\) and \(t \in [a_{n_0}, b_{n_0}]\), we have
\[ d \geq x(t) \geq h(t)\|x\| \geq ch(a_{n_0}) \geq cD_{n_0}. \tag{2.5} \]

Set \(B = \lambda \max\{f(t,s) : a \leq t \leq b, cD_{n_0} \leq s \leq d\}\). Using (2.4) and (2.5), we conclude that
\[ \sup_{x \in P_{c,d}} \lambda \int_a^b g(t)\alpha(t)f(t,x(t))\,dt \leq \lambda \int_{A_{n_0}} g(t)\alpha(t)f(t,x(t))\,dt + \sup_{x \in P_{c,d}} \lambda \int_{[a_{n_0}, b_{n_0}]} g(t)\alpha(t)f(t,x(t))\,dt \leq 1 + B \int_a^b g(t)\alpha(t)\,dt < +\infty, \tag{2.6} \]

that is, (2.3) holds. By Lemma 2.1, \((C_1)\) and \((C_2)\) ensure that \(T_\lambda\) is well defined and is nonnegative. For each \((x,t) \in P_{c,d} \times [a,b]\), by Lemma 2.1 we know that
\[ \|T_\lambda x\| = \sup_{t \in [a,b]} \lambda \int_a^b G(t,s)\alpha(s)f(s,x(s))\,ds \leq \lambda \int_a^b g(s)\alpha(s)f(s,x(s))\,ds \]
and

\[ T_{\lambda}x(t) = \lambda \int_{a}^{b} G(t, s) \alpha(s) f(s, x(s)) \, ds \]

\[ \geq \lambda \int_{a}^{b} h(t) g(s) \alpha(s) f(s, x(s)) \, ds \geq h(t) \| T_{\lambda}x \| , \]

which gives that \( T_{\lambda}x \in P \).

Next we claim that \( T : P_{c,d} \to P \) is completely continuous. It follows from (2.2) and (2.6) that for any \( x \in P_{c,d} \)

\[ \| T_{\lambda}x \| = \sup_{t \in [a, b]} \lambda \int_{a}^{b} G(t, s) \alpha(s) f(s, x(s)) \, ds \leq 1 + B \int_{a}^{b} g(s) \alpha(s) \, ds , \]

that is, the operator \( T_{\lambda} \) is bounded on \( P_{c,d} \). Notice that (C2) implies that for given \( \varepsilon > 0 \), there exists some positive integer \( n > 3 \) such that

\[ \sup_{x \in P_{c,d}} \lambda \int_{A_{n}} g(s) \alpha(s) f(s, x(s)) \, ds < \frac{\varepsilon}{4} , \quad (2.7) \]

It follows from the uniform continuity of Green’s function \( G \) on \([a, b] \times [a, b]\) that there exists some constant \( \delta > 0 \) satisfying

\[ | G(t, s) - G(r, s) | \leq L^{-1} \varepsilon , \quad t, r, s \in [0, 1] \text{ with } | t - r | < \delta , \quad (2.8) \]

where \( L = 2\lambda(b - a)[1 + \max_{w \in [a, b]} \alpha(w) \cdot \max_{(v, w) \in [cD_{n}, d]} f(v, w)] \). By virtue of Lemma 2.1 and lines (2.2), (2.7) and (2.8), we get that

\[ | T_{\lambda}x(t) - T_{\lambda}x(r) | \]

\[ \leq \lambda \int_{a}^{b} | G(t, s) - G(r, s) | \alpha(s) f(s, x(s)) \, ds \]

\[ = \lambda \int_{A_{n}} | G(t, s) - G(r, s) | \alpha(s) f(s, x(s)) \, ds \]

\[ + \lambda \int_{[a_{n}, b_{n}]} | G(t, s) - G(r, s) | \alpha(s) f(s, x(s)) \, ds \]

\[ \leq 2\lambda \int_{A_{n}} g(s) \alpha(s) f(s, x(s)) \, ds + \lambda \int_{[a_{n}, b_{n}]} L^{-1} \varepsilon \alpha(s) f(s, x(s)) \, ds \]

\[ < \varepsilon \]

for all \( (x, t, r) \in P_{c,d} \times [a, b] \times [a, b] \) with \( | t - r | < \varepsilon \), that is, \( \{T_{\lambda}x: x \in P_{c,d}\} \) is equicontinuous on \([a, b]\). Let \( \{x_{n}\}_{n \geq 1} \) be any sequence in \( P_{c,d} \) with \( \lim_{n \to \infty} x_{n} = x \in P_{c,d} \). Condition (C1) implies that for each \( \varepsilon > 0 \) there exists some positive integer \( m_{0} \) satisfying

\[ \sup_{x \in P_{c,d}} \lambda \int_{A_{m_{0}}} g(s) \alpha(s) f(s, x(s)) \, ds < \frac{\varepsilon}{4} . \quad (2.9) \]
Since $f$ is uniformly continuous on $[a_0, b_0] \times [c D_{m_0}, d]$, there exists some positive constant $\delta$ satisfying
\[
|f(v, t) - f(v, r)| < M \varepsilon, \quad v \in [a_0, b_0], \ t, r \in [c D_{m_0}, d] \text{ with } |t - r| < \delta,
\]
where $M = (2\lambda \int_a^b g(s)\alpha(s)\,ds)^{-1}$. From $\lim_{n \to \infty} x_n = x \in P_{c,d}$, we can choose a positive integer $N > m_0$ satisfying $\|x_n - x\| < \delta$ for $n > N$. Clearly,
\[
c D_{m_0} \leq \min\{x_n(t), x(t)\} \leq \max\{x_n(t), x(t)\} \leq d, \quad \forall t \in [a_0, b_0], n \geq 1.
\]
In light of lines (2.9)–(2.11), we infer that for any $n > N$,
\[
\|T_\lambda x_n - T_\lambda x\| = \sup_{t \in [a, b]} \frac{1}{\lambda} \left| \int_a^b G(t, s)\alpha(s)\left[ f(s, x_n(s)) - f(s, x(s))\right] \,ds \right|
\leq \lambda \int_a^b g(s)\alpha(s) \left| f\left(s, x_n(s)\right) - f\left(s, x(s)\right)\right| \,ds
\leq \lambda \int_{A_{m_0}} g(s)\alpha(s) \left| f\left(s, x_n(s)\right) - f\left(s, x(s)\right)\right| \,ds
+ \lambda \int_{[a_0,b_0]} g(s)\alpha(s) \left| f\left(s, x_n(s)\right) - f\left(s, x(s)\right)\right| \,ds
\leq 2\lambda \sup_{u \in P_{c,d}} \int_{A_{m_0}} g(s)\alpha(s) f\left(s, u(s)\right) \,ds + M \lambda \varepsilon \int_{[a_0,b_0]} g(s)\alpha(s) \,ds
< \varepsilon,
\]
which yields that $\lim_{n \to \infty} T_\lambda x_n = T_\lambda x$. That is, $T_\lambda$ is continuous on $P_{c,d}$. It follows from the Arzela–Ascoli theorem that $T_\lambda : P_{c,d} \to P$ is completely continuous. This completes the proof. $\square$

The following fixed-point theorem will be useful in the proof of the existence of positive solutions in the next section.

**Lemma 2.3** (Fixed-Point Theorem of Cone Expansion and Compression Type [12]). Let $(X, \|\cdot\|)$ be a real Banach space and let $Y \subset X$ be a cone in $X$. Assume that $T : Y_{c,d} \to Y$ is a completely continuous operator such that either

(a) $Tu \not\leq u$ for $u \in \partial Y_c$, and $Tu \not\geq u$ for $u \in \partial Y_d$, or
(b) $Tu \not\geq u$ for $u \in \partial Y_c$, and $Tu \not\leq u$ for $u \in \partial Y_d$.

Then $T$ has a fixed point $x \in Y$ with $c < \|x\| < d$.

3. Existence of positive solutions

Now we are ready to establish a few sufficient conditions for the existence of at least one or two nondecreasing positive solutions of the boundary value problem (1.1), (1.2) under suitable
conditions by applying the positivity of Green’s function $G(t, s)$ and the fixed-point theorem of cone expansion and compression type.

**Theorem 3.1.** Assume there exist positive constants $\beta, j, r, i, l$ with $j < i < l$, $r < k$, and $i \geq L(\beta, r)$ such that

\[
\begin{align*}
    f(t, s) > \frac{jm}{\lambda}, & \quad (t, s) \in [p, q] \times [jh(p), j], \\
    f(t, s) < \frac{rs}{\lambda}, & \quad (t, s) \in [a, b] \times [\beta, i], \\
    f(t, s) > \frac{lm}{\lambda}, & \quad (t, s) \in [p, q] \times [lh(p), l].
\end{align*}
\]

If (C1) and (C2) hold, then boundary value problem (1.1), (1.2) possesses at least two nondecreasing positive solutions $x_1, x_2 \in P_j, l$ with $j < \|x_1\| < i < \|x_2\| < l$.

**Proof.** Define the operator $T_\lambda : P_j, i \to P$ by (2.2) with $c = j$ and $d = i$. We now show that

\[
T_\lambda x \not\leq x, \quad x \in \partial P_j.
\]  

Otherwise there exists some $x_0 \in \partial P_j$ with $T_\lambda x_0 \leq x_0$, that is, $x_0 - T_\lambda x_0 \in P$, which implies that $x_0(t) \geq T_\lambda x_0(t)$ for $t \in [a, b]$. Note that

\[
\|x_0\| = j \geq x_0(t) \geq \|x_0\| h(t) \geq jh(p), \quad t \in [p, q].
\]

This, together with Lemma 2.1 and lines (2.2) and (3.1), guarantees that for $t \in [p, q]$

\[
\begin{align*}
    j & \geq x_0(t) \geq T_\lambda x_0(t) = \lambda \int_a^b G(t, s)\alpha(s) f(s, x_0(s)) \, ds \\
    & \geq \lambda \int_{p}^{q} h(t) g(s)\alpha(s) f(s, x_0(s)) \, ds > h(p) jm \int_{p}^{q} g(s)\alpha(s) \, ds = j,
\end{align*}
\]

which is a contradiction. Therefore, (3.4) holds. We next show that

\[
T_\lambda x \not\geq x, \quad x \in \partial P_i.
\]  

Suppose that there exists some $x_1 \in \partial P_i$ with $T_\lambda x_1 \geq x_1$, which gives that $T_\lambda x_1(t) \geq x_1(t)$ for $t \in [a, b]$. Put

\[
B(x_1) = \{ t \in [a, b] : x_1(t) > \beta \} \quad \text{and} \quad \bar{x}_1(t) = \min\{x_1(t), \beta\}, \quad t \in [a, b].
\]

Notice that $\bar{x}_1 \in C([a, b], [0, +\infty))$, $ih(t) \leq x_1(t) \leq \|x_1\| = i$ for $t \in [a, b]$ and there exists some $t_0 \in [a, b]$ satisfying $x_1(t_0) = i$. Consequently, we deduce that $\bar{x}_1(t) = \min\{x_1(t), \beta\} \leq \min\{i, \beta\} = \beta$ for $t \in [a, b]$ and $\bar{x}_1(t_0) = \min\{i, \beta\} = \beta$. Hence $\|\bar{x}_1\| = \beta$. Since

\[
\bar{x}_1(t) = \min\{x_1(t), \beta\} \geq \min\{ih(t), \beta\} \geq h(t)\beta, \quad t \in [a, b],
\]

it follows that $\bar{x}_1 \in \partial P_\beta$. In light of inequality (3.2) and Lemma 2.1, we deduce that

\[
i = \|x_1\| \leq \|T_\lambda x_1\| = \sup_{t \in [a, b]} \lambda \int_a^b G(t, s)\alpha(s) f(s, x_1(s)) \, ds
\]
\[ \leq \lambda \int_a^b g(s)\alpha(s)f(s, x_1(s)) \, ds \]

\[ = \lambda \int_{B(x_1)} g(s)\alpha(s)f(s, x_1(s)) \, ds \]

\[ + \lambda \int_{[a,b]\backslash B(x_1)} g(s)\alpha(s)f(s, x_1(s)) \, ds \]

\[ < ri \int_{B(x_1)} g(s)\alpha(s) \, ds \]

\[ + \lambda \int_{[a,b]} B(x_1) g(s)\alpha(s)ds \]

\[ < \lambda \int_{[a,b]} B(x_1) g(s)\alpha(s)ds \]

\[ + \int_a^b g(s)\alpha(s)ds \]

\[ \leq \lambda \left( \int_{[a,b]} B(x_1) g(s)\alpha(s)ds \right) \]

\[ + \int_a^b g(s)\alpha(s)ds \]

\[ \leq rik^{-1} + M(\beta) \leq i, \]

which is absurd, and hence (3.5) holds. Lemmas 2.2 and 2.3, in conjunction with (3.4) and (3.5), guarantee that the operator \( T_\lambda \) has at least one fixed point \( x_1 \in P_{j,i} \). It is not difficult to verify that

\[ x_1'''(t) = -\lambda \alpha(t)f(t, x_1(t)), \quad a < t < b, \]

\[ x_1(a) = x_1'(a) = x_1'(b) = 0, \]

(3.6)

(3.7)

that is, \( x_1 \) is a solution of problem (1.1), (1.2). Notice that (3.6) implies that \( x_1'''(t) \leq 0 \) for \( t \in (a, b) \). Consequently, \( x_1'' \) is nonincreasing on \([a, b]\). It follows from line (3.7) that \( x_1''(t) \) is nonincreasing on \([a, b]\). This together with \( x_1'(b) = 0 \) means that \( x_1'(t) \geq 0 \) for \( t \in [a, b] \), which yields that \( x_1 \) is nondecreasing on \([a, b]\). It follows from Lemma 2.1 and \( x_1 \in P_{j,i} \) that \( x_1(t) \geq h(t)\|x_1\| > 0 \) for \( t \in (a, b) \). Consequently, \( x_1 \) is a nondecreasing positive solution of (1.1), (1.2) in \( P_{j,i} \).

Define the operator \( T_\lambda : P_{j,l} \to P \) by (2.2) with \( c = i \) and \( d = l \). Similarly, we can show that boundary value problem (1.1), (1.2) has at least one nondecreasing positive solution \( x_2 \in P_{j,l} \).

This completes the proof. \( \Box \)

**Theorem 3.2.** Assume that there exist positive constants \( \beta, r \) and \( i \) with \( r < k \), \( i \geq L(\beta, r) \) satisfying (3.2) and

\[ \min\{f_0, f_\infty\} > \frac{m}{\lambda h(p)}. \]  

(3.8)

If \((C_1)\) and \((C_2)\) hold, then there exist two positive constants \( j \) and \( l \) with \( j < \beta \) and \( l > \frac{i}{h(p)} \) such that (1.1), (1.2) possesses at least two nondecreasing positive solutions \( x_1, x_2 \in P \) with \( j < \|x_1\| < i < \|x_2\| < l \).

**Proof.** Let \( \varepsilon_1 = \frac{1}{2}(f_0 - \frac{m}{\lambda h(p)}) \) and \( \varepsilon_2 = \frac{1}{2}(f_\infty - \frac{m}{\lambda h(p)}) \). In view of (3.8), we can choose sufficiently small \( j \in (0, \beta) \) and sufficiently large \( l > \frac{i}{h(p)} \) such that

\[ f(t, s) \geq s(f_0 - \varepsilon_1) > \frac{sm}{\lambda h(p)}, \quad (t, s) \in [p, q] \times (0, j], \]

and

\[ f(t, s) \geq s(f_\infty - \varepsilon_2) > \frac{sm}{\lambda h(p)}, \quad (t, s) \in [p, q] \times [lh(p), +\infty], \]
which imply that (3.1) and (3.3) are fulfilled, respectively. Thus Theorem 3.2 follows from Theorem 3.1. This completes the proof.

**Theorem 3.3.** Assume that there exist positive constants $\beta$, $r$ and $i$ with $r < k$, $i \geq L(\beta, r)$ satisfying (3.2) and
\[
\min\{f_0, f_\infty\} = +\infty. \tag{3.9}
\]
If (C$_1$) and (C$_2$) hold, then there exist two positive constants $j$ and $l$ with $j < \beta$ and $l > \frac{i}{h(p)}$ such that Eq. (1.1), (1.2) possesses at least two nondecreasing positive solutions $x_1, x_2 \in P$ with $j < \|x_1\| < i < \|x_2\| < l$.

**Proof.** Notice that (3.9) ensures that there exist sufficiently small $j \in (0, \beta)$ and sufficiently large $l > \frac{i}{h(p)}$ such that
\[
f(t, s) > \frac{sm}{\lambda}, \quad (t, s) \in [p, q] \times (0, j],
\]
and
\[
f(t, s) > \frac{sm}{\lambda}, \quad (t, s) \in [p, q] \times [lh(p), +\infty),
\]
which yield that (3.1) and (3.3) hold. Thus Theorem 3.3 follows from Theorem 3.1. This completes the proof.

The proofs of Theorems 3.4 and 3.5 are quite similar to those of Theorems 3.1 and 3.2, respectively, and will be omitted.

**Theorem 3.4.** Assume that there exist positive constants $r, j, \tau, \beta, i, \gamma$ and $l$ with $\max\{r, \tau\} < k$, $i < j < l$, $i \geq L(\beta, r)$ and $l \geq L(\gamma, \tau)$ satisfying (3.1), (3.2) and
\[
f(t, s) < \frac{\tau s}{\lambda}, \quad (t, s) \in [a, b] \times [\gamma, l]. \tag{3.10}
\]
If (C$_1$) and (C$_2$) hold, then boundary value problem (1.1), (1.2) possesses at least two nondecreasing positive solutions $x_1, x_2 \in P_{i,l}$ with $i < \|x_1\| < j < \|x_2\| < l$.

**Theorem 3.5.** Assume that there exist positive constants $r, j, \tau, \beta$ and $i$ with $\max\{r, \tau\} < k$ and $j > i \geq L(\beta, r)$ satisfying (3.1), (3.2) and
\[
\mathcal{f} \sim < \frac{\tau}{\lambda}. \tag{3.11}
\]
If (C$_1$) and (C$_2$) hold, then (1.1), (1.2) possesses at least two nondecreasing positive solutions $x_1, x_2 \in P$ with $i < \|x_1\| < j < \|x_2\|$.

Now we study the existence of at least one positive solution for (1.1), (1.2).

**Theorem 3.6.** Let the function $f : [a, b] \times (0, +\infty) \to [0, +\infty)$ satisfy the following
\[
f_0 = +\infty \quad \text{and} \quad f^\infty = 0. \tag{3.12}
\]
If (C$_1$) and (C$_2$) hold, then for any $\lambda > 0$ boundary value problem (1.1), (1.2) possesses a nondecreasing positive solution in $P$. 
Proof. Let $\lambda$ be an arbitrary positive constant. It follows from (3.12) that there exist $j > 0$ and $\beta > j$ satisfying

$$f(t, s) > \frac{sm}{\lambda h(p)}, \quad (t, s) \in [p, q] \times (0, j],$$

and

$$f(t, s) < \frac{ks}{2\lambda} \quad \text{for} \quad (t, s) \in [a, b] \times [\beta, +\infty),$$

which yield that (3.1) and (3.2) hold with $r = \frac{k}{2}$ and $i = L(\beta, r)$. As in the proof of Theorem 3.1, we conclude that (1.1), (1.2) possesses a nondecreasing positive solution in $P$. This completes the proof. □

Theorem 3.7. Let $(C_1)$ and $(C_2)$ hold. Assume that

$$f_0 = +\infty \quad \text{and} \quad \bar{f} \infty > 0. \quad (3.13)$$

Then for each $\lambda \in (0, \frac{k}{\bar{f} \infty})$, boundary value problem (1.1), (1.2) possesses a nondecreasing positive solution in $P$.

Proof. Let $\lambda \in (0, \frac{k}{\bar{f} \infty})$ and $r = \frac{1}{2}(\lambda \bar{f} \infty + k)$. As in the proof of Theorem 3.4, we infer that (3.1) holds for some $j > 0$. Since $\bar{f} \infty < \frac{f_0}{\lambda}$, it follows that for sufficiently small $\epsilon > 0$ there exists some $\beta > j$ satisfying

$$f(t, s) < \frac{rs}{\lambda + \epsilon} \leq \frac{rs}{\lambda}, \quad (t, s) \in [a, b] \times [\beta, +\infty),$$

which implies that (3.2) holds for $i = L(\beta, r)$. It follows from the proof of Theorem 3.1 that (1.1), (1.2) possesses a nondecreasing positive solution in $P$. This completes the proof. □

Theorem 3.8. Let $(C_1)$ and $(C_2)$ hold. Assume that

$$0 < m \bar{f} \infty < k h(p) f_0. \quad (3.14)$$

Then for any $\lambda \in (\frac{m}{h(p)f_0}, \frac{k}{\bar{f} \infty})$, boundary value problem (1.1), (1.2) possesses a nondecreasing positive solution in $P$.

Proof. Let $\lambda \in (\frac{m}{h(p)f_0}, \frac{k}{\bar{f} \infty})$. Choose $\epsilon_1 = \frac{1}{2}(f_0 - \frac{m}{\lambda})$ and $\epsilon_2 = \frac{1}{2}(\frac{k}{\lambda} - \bar{f} \infty)$. It follows that there exist positive constants $j < \beta$ and $r = \frac{k + \lambda \bar{f} \infty}{2}$ satisfying

$$f(t, s) > (f_0 - \epsilon_1)s = \frac{1}{2}\left(f_0 + \frac{m}{\lambda}\right)s = \frac{ms}{\lambda h(p)}, \quad (t, s) \in [p, q] \times (0, j],$$

and

$$f(t, s) < (\bar{f} \infty + \epsilon_2)s = \frac{1}{2}\left(\frac{k}{\lambda} + \bar{f} \infty\right)s = \frac{rs}{\lambda}, \quad (t, s) \in [a, b] \times [\beta, +\infty),$$

which imply that (3.1) and (3.2) hold for $i = L(\beta, r)$. As in the proof of Theorem 3.1, we conclude immediately that (1.1), (1.2) possesses a nondecreasing positive solution in $P_{j, i}$. This completes the proof. □
4. Applications

In this section, we construct four examples to illustrate the applicability of the results presented in Section 3.

Example 4.1. The singular autonomous equation

\[ x'''(t) + \frac{1}{\sqrt{x(t)}} + \frac{x^2(t)}{2N} = 0, \quad t \in (0, 1), \quad (4.1) \]

under boundary conditions

\[ x(0) = x''(0) = x'(1) = 0 \quad (4.2) \]

possesses at least two nondecreasing positive solutions in \( P = \{ x \in C([0, 1], \mathbb{R}): x(t) \geq h(t) \|x\|, t \in [0, 1] \} \), where \( N \) is a sufficiently large constant and \( h(t) = \frac{t}{2}, \ t \in [0, 1] \). In fact, define the functions \( f : (0, +\infty) \rightarrow \mathbb{R}^+ \) and \( \alpha : [0, 1] \rightarrow \mathbb{R}^+ \) by

\[ f(s) = \frac{1}{\sqrt{s}} + \frac{s^2}{2N}, \quad s \in (0, +\infty), \]

and

\[ \alpha(t) = 1, \quad t \in [0, 1], \]

where \( N > L(1, \frac{3}{2}) \). Let \( p = \frac{1}{4}, \ q = \frac{1}{2}, \ a = 0, \ b = 1, \ \lambda = 1, \ \beta = 1, \ r = \frac{3}{2}, \ i = L(1, \frac{3}{2}), \) \( \{a_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \) be strictly decreasing and strictly increasing sequences, respectively, with \( \lim_{n \to \infty} a_n = 0, \lim_{n \to \infty} b_n = 1 \) and \( a_1 < b_1 \). It is not difficult to verify that

\[ g(t) = 1 - t, \quad t \in [0, 1], \]

\[ k = \left[ \int_a^b g(t)\alpha(t)\,dt \right]^{-1} = 2, \]

\[ m = \left[ \int_p^q g(t)\alpha(t)\,dt \right]^{-1} = \frac{256}{5}, \]

\[ 0 < \int_p^q g(t)\alpha(t)\,dt = \frac{5}{32}, \quad \int_a^b g(t)\alpha(t)\,dt = \frac{1}{2} < +\infty, \]

\[ f_0 = f_\infty = +\infty \]

and

\[ f(s) = \frac{1}{\sqrt{s}} + \frac{s^2}{2N} < s + \frac{s}{2} = \frac{3}{2}s, \quad s \in [\beta, i]. \]

For any \( d > c > 0 \), we infer that

\[ \sup_{x \in P_{c,d}} \int_{\lambda_n} g(t)\alpha(t)f(x(t))\,dt \leq \int_{\lambda_n} \left[ \sqrt{\frac{2}{c\sqrt{t}}} + \frac{d^2}{2N} \right] \,dt \to 0 \quad \text{as} \ n \to \infty. \]

Therefore the assumptions of Theorem 3.3 are satisfied. Thus Theorem 3.3 ensures that there exists \( j \in (0, 1) \) and finite \( l > 64 \) such that Eq. (4.1), (4.2) possesses at least two nondecreasing positive solutions \( x_1, x_2 \in P \) with \( j < x_1(1) < 8 < x_2(1) < l \).
Example 4.2. Consider the singular nonlinear third-order differential equation
\[
x'''(t) + \frac{\lambda}{(b-t)(t-a)^2} \left[ \sqrt{(b-t)}(t-a) + \frac{(b-t)^2 \sqrt{1-a}}{\sqrt{x(t)}} \right] + \sin^2 \frac{1}{(x(t))^2} + \sqrt{x(t)} |\ln x(t)| = 0, \quad a < t < b,
\]
with boundary conditions (1.2). Define the functions \( \alpha : (a, b) \to [0, +\infty) \) and \( f : [a, b] \times (0, +\infty) \to [0, +\infty) \) by
\[
\alpha(t) = \frac{1}{(b-t)\sqrt{t-a}}, \quad t \in (a, b),
\]
\[
f(t, s) = \sqrt{(b-t)(t-a)} + \frac{(b-t)^2 \sqrt{t-a}}{\sqrt{s}} + \sin^2 \frac{1}{s^2} + \sqrt{s} |\ln s|,
\]
\((t, s) \in [a, b] \times (0, +\infty)\).

It is easy to verify that the assumptions of Theorem 3.6 are fulfilled. Thus Theorem 3.6 ensures that boundary value problem (4.3), (1.2) possesses a nondecreasing positive solution in \( P \) for any \( \lambda > 0 \).

Example 4.3. Consider the singular nonlinear third-order differential equation
\[
x'''(t) + \frac{\lambda}{(b-a)(b-t)(t-a)^2} \left( \frac{t-a}{\sqrt{|x(t)|}} + x(t) \right) = 0, \quad a < t < b,
\]
with boundary conditions (1.2). Define the functions \( \alpha : (a, b) \to [0, +\infty) \) and \( f : [a, b] \times (0, +\infty) \to [0, +\infty) \) by
\[
\alpha(t) = \frac{1}{(b-a)(b-t)\sqrt{t-a}}, \quad t \in (a, b),
\]
\[
f(t, s) = \frac{t-a}{\sqrt{s}} + s, \quad (t, s) \in [a, b] \times (0, +\infty).
\]
It is clear that the assumptions of Theorem 3.7 are fulfilled. Thus Theorem 3.7 ensures that for any \( \lambda \in (0, \frac{1}{2\sqrt{b-a}}) \), boundary value problem (4.4), (1.2) possesses a nondecreasing positive solution in \( P \).

Example 4.4. Consider the singular nonlinear third-order differential equation
\[
x'''(t) + \frac{\lambda}{(b-a)(b-t)(t-a)^2} \left[ \frac{(t-p)^2}{\sqrt{|x(t)|}} + x(t) \left( 1 + \frac{A + x(t)}{1 + x^2(t)} \right) \right] = 0, \quad a < t < b,
\]
with boundary conditions (1.2), where \( A = \frac{8 \sqrt{(b-a)}}{(p-a)(\sqrt{q-a} - \sqrt{p-a})} \). Define the functions \( \alpha : (a, b) \to [0, +\infty) \) and \( f : [a, b] \times (0, +\infty) \to [0, +\infty) \) by
\[
\alpha(t) = \frac{1}{(b-a)(b-t)\sqrt{(t-a)^2}}, \quad t \in (a, b),
\]
\[
f(t, s) = \frac{(t-p)^2}{\sqrt{s}} + s \left( 1 + \frac{A+s}{1+s^2} \right), \quad (t, s) \in [a, b] \times (0, +\infty).
Obviously the conditions of Theorem 3.8 are satisfied. Consequently Theorem 3.8 gives that for any \( \lambda \in \left( \frac{4(b-a)^2}{(p-a)^2 \left( \frac{2}{3(q-a)} - \frac{1}{3(p-a)} \right) + 24 \left( \frac{1}{3(b-a)} \right)^{7/3}} \right), \) boundary value problem (4.5), (1.2) possesses a nondecreasing positive solution in \( P \).

**Remark 4.5.** Theorem 4 in [10] is not applicable for the boundary value problems in Examples 4.1–4.4 because of the singularities in these boundary value problems.

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**References**


