Abstract

In this study, linear second-order delta-nabla matrix equations on time scales are shown to be formally self-adjoint equations with respect to a certain inner product and the associated self-adjoint boundary conditions. After a connection is made with symplectic dynamic systems on time scales, we introduce a generalized Wronskian and establish a Lagrange identity and Abel’s formula. Two reduction-of-order theorems are given. Solutions of the second-order self-adjoint equation are then shown to be related to corresponding solutions of a first-order Riccati equation. The study is concluded with a comprehensive roundabout theorem relating key equivalences.

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1 Introduction

Dynamic equations on time scales have been introduced by Aulbach and Hilger [7, 14] to unify and extend the theory of ordinary differential equations, difference equations, quantum equations, and all other differential systems defined over nonempty closed subsets of the real line. Already several important problems concerning higher-order scalar dynamic equations on time scales, involving
only delta differentiation, have been developed [9, 10, 11]. In [6] and [17] second-order self-adjoint linear dynamic equations on time scales were introduced and examined by making use of both delta and nabla derivatives. Quite recently [5], two classes of higher-order dynamic equations on time scales, involving both delta and nabla derivatives simultaneously, were shown to be (formally) self-adjoint in the classic sense provided certain self-adjoint boundary conditions were satisfied. In this present work we aim to extend these notions to systems by providing an analysis of the second-order (delta nabla) matrix dynamic equation
\[(PX^\Delta)^\nabla (t) + Q(t)X(t) = 0, \quad (1.1)\]
which will be shown to be associated with a self-adjoint operator with respect to arbitrary time scales, including continuous, discrete, and quantum time scales as corollaries. More commonly authors [1, 2, 3], [10, Chapter 5], [13] focus on
\[(PX^\Delta)^\Delta (t) + Q(t)x^\sigma (t) = 0, \quad (1.2)\]
an equation they often dub the “self-adjoint” equation because it admits a Lagrange identity. The self-adjoint form (1.1), however, is an appropriate generalization and extension of the classic self-adjoint form from ordinary differential equations [12, 16, 18, 19, 20, 21]
\[(PX')' (t) + Q(t)X (t) = 0, \quad (1.3)\]
and the discrete version [4, 15]
\[\Delta (P(t)\Delta X (t - 1)) + Q(t)X(t) = 0,\]
to dynamic equations on time scales.

The paper is constructed as follows. In Section 2 we explore (1.1), show how it is formally a self-adjoint equation, introduce a generalized Wronskian and establish a Lagrange identity and Abel’s formula. Section 3 contains a result allowing the switch from delta to nabla integrals, and two reduction of order theorems. A corresponding Riccati matrix equation is introduced in Section 4, the solutions of which are related back to solutions of (1.1). Section 5 explores a related quadratic functional and a foundational result, Picone’s identity. A full connection among the various aspects of the paper is established in Section 6 in a Reid roundabout theorem. A brief overview of time scales is provided in Section 7.

2 Self-Adjoint Matrix Equations

Let \(P \) and \(Q \) be Hermitian \(n \times n\)-matrix-valued functions on a time scale \(\mathbb{T} \) such that \(P > 0 \) (positive definite) and \(Q \) are continuous for all \(t \in \mathbb{T} \). (A matrix \(M \) is Hermitian iff \(M^* = M \), where \(^* \) indicates conjugate transpose.) In this section we are concerned with the second-order (formally) self-adjoint matrix dynamic equation
\[LX = 0, \text{ where } LX(t) := (PX^\Delta)^\nabla (t) + Q(t)X(t), \quad t \in \mathbb{T}^\kappa_n. \quad (2.1)\]
Definition 2.1. Let $\mathbb{D}$ denote the set of all $n \times n$ matrix-valued functions $X$ defined on $\mathbb{T}$ such that $X^\Delta$ is continuous on $\mathbb{T}^c$ and $(PX^\Delta)\nabla$ is left-dense continuous on $\mathbb{T}^c_\kappa$. Then $X$ is a solution of (2.1) on $\mathbb{T}$ provided $X \in \mathbb{D}$ and

$$LX(t) = 0 \quad \text{for all} \quad t \in \mathbb{T}^c_\kappa.$$ 

Remark 2.2. Note that if $X \in \mathbb{D}$, then $(PX^\Delta)\nabla$ is left-dense continuous on $\mathbb{T}^c_\kappa$, but if $X$ is a solution of $LX = 0$, then $(PX^\Delta)\nabla$ is continuous on $\mathbb{T}^c_\kappa$ by the continuity of $Q$.

Lemma 2.3. Let $X \in \mathbb{D}$. If we set $Z := \begin{pmatrix} X \\ PX^\Delta \end{pmatrix}$ and $S := \begin{pmatrix} 0 & P^{-1} \\ -Q^\sigma & -\mu Q^\sigma P^{-1} \end{pmatrix}$, then $X$ solves (2.1) on $\mathbb{T}^c_\kappa$ if and only if $Z$ solves

$$Z^\Delta = S(t)Z, \quad t \in \mathbb{T}^c.$$ 

Proof. Let $X \in \mathbb{D}$ and $Z, S$ be given as in (2.2); by Theorem 7.1,

$$SZ = \begin{pmatrix} X^\Delta \\ (PX^\Delta)^\Delta \end{pmatrix}.$$ 

If $Z$ solves (2.3), then $Z^\Delta = (PX^\Delta)^\Delta$ exists, is right-dense continuous, and $(PX^\Delta)^\Delta = -Q^\sigma X^\sigma$ on $\mathbb{T}^c$. Consider Theorem 7.4 (i) and set $A$ as in (7.1). For $t \in \mathbb{T}^c_\kappa \cap (\mathbb{T} \setminus A)$, $\sigma(\rho(t)) = t$ and

$$(PX^\Delta)^\nabla = (PX^\Delta)^\Delta \nu = -Q^\sigma \sigma^\nu = -QX,$$

so that $X$ solves $LX = 0$ on $\mathbb{T}^c_\kappa \cap (\mathbb{T} \setminus A)$. For $t \in \mathbb{T}^c_\kappa \cap A$, $t$ is a left-dense right-scattered point, whence $\lim_{s \to t^-} \sigma(s) = t$ by Definition 7.3. From this we obtain

$$(PX^\Delta)^\nabla(t) = \lim_{s \to t^-} (PX^\Delta)^\Delta(s) = - \lim_{s \to t^-} Q^\sigma \sigma^\nu(s) X^\sigma(s) = -Q(t)X(t),$$

as $Q$ is (left-dense) continuous. Therefore, $X$ solves $LX = 0$ on $\mathbb{T}^c_\kappa \cap A$ as well. If $X$ solves (2.1), then $(PX^\Delta)^\nabla = -QX$ on $\mathbb{T}^c_\kappa$. Using Theorem 7.4 (ii), the continuity of $X$ and the (right-dense) continuity of $Q$ we have

$$(PX^\Delta)^\Delta = (PX^\Delta)^\nabla \sigma = -Q^\sigma X^\sigma.$$ 

Thus, $Z^\Delta = SZ$ on $\mathbb{T}^c$. \qed

Definition 2.4 (Regressivity). An $n \times n$ matrix-valued function $M$ on a time scale $\mathbb{T}$ is regressive with respect to $\mathbb{T}$ provided

$$I + \mu(t)M(t) \text{ is invertible for all } t \in \mathbb{T}^c,$$

and the class of all such regressive and rd-continuous functions is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n \times n}).$$
Definition 2.5. An \( n \times n \) matrix-valued function \( A \) on a time scale \( \mathbb{T} \) is called \( \nu \)-regressive (with respect to \( \mathbb{T} \)) provided
\[
I - \nu(t)A(t) \text{ is invertible for all } t \in \mathbb{T},
\]
and the class of all such \( \nu \)-regressive and \( \ell d \)-continuous functions is denoted by
\[
\mathcal{R}_\nu = \mathcal{R}_\nu(\mathbb{T}) = \mathcal{R}_\nu(\mathbb{T}, \mathbb{R}^{n \times n}).
\]

Lemma 2.6. Let \( A : \mathbb{T} \to \mathbb{R}^n \). Then \( A \) is regressive if and only if \( -A^\rho \) is \( \nu \)-regressive, and \( 1 + \nu(t)A(t) > 0 \) for all \( t \in \mathbb{T} \) if and only if \( 1 - \nu(t)B^\rho(t) > 0 \) for all \( t \in \mathbb{T} \). Similarly, if \( B : \mathbb{T} \to \mathbb{R} \), then \( B \) is \( \nu \)-regressive if and only if \( -B^\sigma \) is regressive, and \( 1 - \nu(t)B(t) > 0 \) for all \( t \in \mathbb{T} \) if and only if \( 1 - \mu(t)B^\sigma(t) > 0 \) for all \( t \in \mathbb{T} \).

Theorem 2.7. Let \( a \in \mathbb{T}^\kappa \) be fixed and \( X_a, X^\Delta_a \) be given constant \( n \times n \) matrices. Then the initial value problem
\[
PX^\Delta + Q(t)X(t) = 0, \quad X(a) = X_a, \quad X^\Delta(a) = X^\Delta_a
\]
has a unique solution \( X : \mathbb{T} \to \mathbb{R}^n \).

Proof. For \( S \) given in (2.2), \( S \in C_{rd} \) is regressive by Definition 2.4 on \( \mathbb{T} \) since
\[
(I + \mu S)^{-1} = \begin{pmatrix} I & \mu P^{-1} \\ -\mu Q^\sigma & I - \mu^2 Q^\sigma P^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} I & -\mu P^{-1} \\ \mu Q^\sigma & I \end{pmatrix}
\]
exists on \( \mathbb{T}^\kappa \). Hence, by [10, Theorem 5.8], any initial value problem
\[
Z^\Delta = S(t)Z, \quad Z(a) = Z_a,
\]
in other words,
\[
LX = 0, \quad X(a) = X_a, \quad X^\Delta(a) = X^\Delta_a
\]
with any choice of a \( 2n \times n \)-matrix \( Z_a \), i.e., \( n \times n \)-matrices \( X_a \) and \( X^\Delta_a \), has a unique solution.

In view of the theorem just proven, the following definition is now possible.

Definition 2.8. The unique solution of the initial value problem
\[
LX = 0, \quad X(a) = 0, \quad X^\Delta(a) = P^{-1}(a)
\]
is called the principal solution of (2.1) \( (at a) \), while the unique solution of the initial value problem
\[
LX = 0, \quad X(a) = -I, \quad X^\Delta(a) = 0
\]
is called the associated (coprincipal) solution of (2.1) \( (at a) \).

The following definition is standard; see, for example, [10, Definition 7.7].
Definition 2.9. For the $2n \times 2n$ matrix $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, the $2n \times 2n$ matrix $S \in \mathbb{C}^{rd}$ is symplectic with respect to $T$ iff
\[ S^*J + JS + \mu S^*JS = 0, \quad t \in T, \]
and the $2n \times 2n$ matrix $A$ is symplectic iff
\[ A^*JA = J. \]

Lemma 2.10. The matrix $S$ associated with (2.1) given in (2.2) is symplectic with respect to $T$, $I + \mu S$ is symplectic, and $S$ is regressive with respect to $T$.

Proof. The proof is a straightforward calculation; see also [10, Theorem 7.11].

Remark 2.11. Note that we have used the assumption that the coefficient matrix $Q$ in (2.1) is continuous to make a connection with symplectic systems to more directly obtain existence and uniqueness of solutions to initial value problems, as in Theorem 2.7. This assumption would also allow us to take sigma of both sides of (1.1) to get the form (1.2), making (2.1) more general. It is possible to only require that $Q$ is left-dense continuous, but then more work is needed to prove the existence and uniqueness of solutions to initial value problems involving (2.1).

Definition 2.12. If $X, Y \in \mathbb{D}$, then the (generalized) Wronskian matrix of $X$ and $Y$ is given by
\[ W(X, Y)(t) = X^*(t)P(t)Y^\Delta(t) - [P(t)X^\Delta(t)]^*Y(t) \]
for $t \in T^\kappa$.

Theorem 2.13 (Lagrange Identity). If $X, Y \in \mathbb{D}$, then
\[ W(X, Y)^\nabla(t) = X^*(t)(LY)(t) - (LX(t))^*Y(t), \quad t \in T^\kappa. \]

Proof. For $X, Y \in \mathbb{D}$, using the product rule for nabla derivatives we have
\[
W(X, Y)^\nabla = \{X^*PY^\Delta - (PX^\Delta)^*Y\}^\nabla \\
\overset{\text{Thm 7.4}}{=} X^*(PY^\Delta)^\nabla + (X^*)^\nabla P^\rho Y^\nabla - (PX^\Delta)^*^\nabla Y - (PX^\Delta)^*^\nabla Y \\
= X^*(LY - QY) + X^*^\nabla P^\rho Y^\nabla - X^*^\nabla P^\rho Y^\nabla \\
- (LX - QX)^*Y \\
= X^*(LY) - (LX)^*Y
\]
on $T^\kappa$.

Definition 2.14. Define the inner product of $n \times n$ matrices $M$ and $N$ on $[a, b]$ to be
\[ (M, N) = \int_a^b M^*(t)N(t)\nabla t, \quad M, N \in C_{id}(T), \quad a, b \in T^\kappa. \] (2.5)
For differential equations, $\mathbb{T} = [a, b]$, this is the classic definition
\[ \langle M, N \rangle = \int_a^b M^*(t)N(t)\,dt, \quad M, N \in C(\mathbb{R}), \quad a, b \in \mathbb{R}. \]

In the case of difference equations, $\mathbb{T} = [a, b]_\mathbb{Z}$, this becomes
\[ \langle M, N \rangle = \sum_{t=a+1}^{b} M^*(t)N(t), \quad a, b \in \mathbb{Z}. \]

Finally, for quantum equations, $\mathbb{T} = [q^a_q, q^b_q]_\mathbb{Q}$ for $q > 1$, the inner product becomes a Jackson integral, given here by
\[ \langle M, N \rangle = (q-1) \sum_{t=a+1}^{b} q^{t-1}M^*(q^t)N(q^t), \quad q^a_q, q^b_q \in q^\mathbb{Z}. \]

**Corollary 2.15 (Self-Adjoint Operator).** The operator $L$ in (2.1) is formally self-adjoint with respect to the inner product (2.5); that is, the identity
\[ \langle LX, Y \rangle = \langle X, LY \rangle \]

holds provided $X, Y \in \mathbb{D}$ and $X, Y$ satisfy $W(X, Y)(t)|^b_a = 0$, called the self-adjoint boundary conditions.

**Proof.** Let $X, Y \in \mathbb{D}$, and $X, Y$ satisfy $W(X, Y)(t)|^b_a = 0$. From Definition 2.12 and Theorem 2.13 we see that Green’s formula holds, namely
\[ \int_a^b W(X, Y)\nabla(t)\nabla t = W(X, Y)(t)|^b_a = \langle X, LY \rangle - \langle LX, Y \rangle. \]

Another immediate corollary of the Lagrange identity is Abel’s matrix formula. This and the following three results may be proven in a way analogous to their proofs in the case of $\mathbb{T} = \mathbb{R}$.

**Corollary 2.16 (Abel’s Formula).** If $X, Y$ are solutions of (2.1) on $\mathbb{T}$, then
\[ W(X, Y)(t) \equiv C, \quad t \in \mathbb{T}_\kappa, \]

where $C$ is a constant matrix.

**Corollary 2.17.** If $X, Y$ are solutions of (2.1) on $\mathbb{T}$, then either $W(X, Y)(t) = 0$ for all $t \in \mathbb{T}^\kappa$, or $W(X, Y)(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$.

**Theorem 2.18.** Any two prepared solutions of (2.1) on $\mathbb{T}$ are linearly independent iff their Wronskian is nonzero.
Theorem 2.19. Equation (2.1) on \( \mathbb{T} \) has two linearly independent solutions, and every solution of (2.1) on \( \mathbb{T} \) is a linear combination of these two solutions.

Theorem 2.20 (Converse of Abel’s Formula). Assume \( X \) is a solution of (2.1) on \( \mathbb{T} \) such that \( X^{-1} \) exists on \( \mathbb{T} \). If \( Y \in \mathbb{D} \) satisfies \( W(X,Y)(t) \equiv C \), where \( C \) is a constant matrix, then \( Y \) is also a solution of (2.1).

Proof. Suppose that \( X \) is a solution of (2.1) such that \( X^{-1} \) exists on \( \mathbb{T} \), and assume \( Y \in \mathbb{D} \) satisfies \( W(X,Y)(t) \equiv C \), where \( C \) is a constant matrix. By the Lagrange identity (Theorem 2.13) we have

\[
0 \equiv W(X,Y)^\nabla(t) = X^\ast(t)(LY)(t) - (LX)(t)Y^\ast(t) = X^\ast(t)(LY)(t), \quad t \in \mathbb{T}_\kappa^c.
\]

As \( (X^\ast)^{-1} \) exists on \( \mathbb{T} \), \( (LY)(t) = 0 \) on \( \mathbb{T}_\kappa^c \). Thus \( Y \) is also a solution of (2.1). \( \square \)

From Abel’s formula we get that if \( X \in \mathbb{D} \) is a solution of (2.1) on \( \mathbb{T} \), then

\[
W(X,X)(t) \equiv C, \quad t \in \mathbb{T}_\kappa^c,
\]

where \( C \) is a constant matrix. With this in mind we make the following definition.

Definition 2.21. Let \( X, Y \in \mathbb{D} \) and \( W \) be given as in (2.12).

(i) \( X \in \mathbb{D} \) is a prepared (conjoined, isotropic) solution of (2.1) iff \( X \) is a solution of (2.1) and

\[
W(X,X)(t) \equiv 0, \quad t \in \mathbb{T}^c.
\]

(ii) \( X, Y \in \mathbb{D} \) are normalized prepared bases of (2.1) iff \( X, Y \) are two prepared solutions of (2.1) with

\[
W(X,Y)(t) \equiv I, \quad t \in \mathbb{T}^c.
\]

Theorem 2.22. Assume that \( X \in \mathbb{D} \) is a solution of (2.1) on \( \mathbb{T} \). Then the following are equivalent:

(i) \( X \) is a prepared solution;

(ii) \( X^\ast(t)P(t)X^\Delta(t) \) is Hermitian for all \( t \in \mathbb{T}^c \);

(iii) \( X^\ast(t_0)P(t_0)X^\Delta(t_0) \) is Hermitian for some \( t_0 \in \mathbb{T}^c \).

Proof. Use the Wronskian \( W \) and Abel’s formula. \( \square \)

Note that one can easily get prepared solutions of (2.1) by taking initial conditions at \( t_0 \in \mathbb{T} \) so that \( X^\ast(t_0)P(t_0)X^\Delta(t_0) \) is Hermitian.

In the Sturmian theory for (2.1) the matrix function \( X^\ast P X^\circ \) is important. We note the following result.
Lemma 2.23. Let $X$ be a solution of (2.1). If $X$ is prepared, then

$$X^*(t)P(t)X^\sigma(t)$$

is Hermitian for all $t \in T^\kappa$.

Conversely, if there is $t_0 \in T^\kappa$ such that $\mu(t_0) > 0$ and $X^*(t_0)P(t_0)X^\sigma(t_0)$ is Hermitian, then $X$ is a prepared solution of (2.1). Moreover, if $X$ is an invertible prepared solution, then

$$P(t)X^\sigma(t)X^{-1}(t), P(t)X(t)(X^\sigma)^{-1}(t), \text{ and } Z(t) := P(t)X^\Delta(t)X^{-1}(t)$$

are Hermitian for all $t \in T^\kappa$.

Proof. Let $X$ be a solution of (2.1). Then $X$ is delta differentiable on $T^\kappa$, and the relation (apply the matrix version of Theorem 7.1)

$$X^*PX^\sigma = X^*P(X + \mu X^\Delta) = X^*PX + \mu X^*PX^\Delta$$

proves the first two statements of this lemma. Now assume that $X$ is an invertible prepared solution of (2.1). Then

$$X^*PX^\sigma = (X^\sigma)^*PX \quad \text{and} \quad X^*PX^\Delta = (X^\Delta)^*PX$$

(2.6)

on $T^\kappa$ by Theorem 2.22 and what we proved earlier. We multiply the first equation in (2.6) from the left with $(X^{-1})^*$ and from the right with $X^{-1}$ to obtain that $PX^\sigma X^{-1}$ is Hermitian. To see that $PX(X^\sigma)^{-1}$ is Hermitian, we multiply the first equation in (2.6) with $(X^\sigma)^{-1}$ from the left and with $(X^\sigma)^{-1}$ from the right. Multiplying the second equation in (2.6) with $(X^{-1})^*$ from the left and with $X^{-1}$ from the right shows that $Z$ is Hermitian. \(\square\)

Lemma 2.24. Assume that $X$ is a prepared solution of (2.1) on $T$. Then the following are equivalent:

(i) $(X^*)^\sigma PX = X^*PX^\sigma > 0$ on $T^\kappa$;

(ii) $X$ is invertible and $PX^\sigma X^{-1} > 0$ on $T^\kappa$;

(iii) $X$ is invertible and $PX(X^\sigma)^{-1} > 0$ on $T^\kappa$.

Proof. First note that $X^*(\sigma(t))P(t)X(t) > 0$ for $t \in T^\kappa$ implies that $X(t)$ is invertible for $t \in T$. Since $X$ is a prepared solution of (2.1), by Lemma 2.23 we have

$$PX^\sigma X^{-1} = (X^{-1})^*X^\sigma P \quad \text{and} \quad PX(X^\sigma)^{-1} = ((X^\sigma)^{-1})^*X^\sigma P$$

(2.7)

on $T^\kappa$. We multiply the right-hand side of the first equation in (2.7) from the right with $XX^{-1}$ to obtain the equivalence of (i) and (ii). For the equivalence of (i) and (iii), multiply the right-hand side of the second equation in (2.7) from the right with $X^\sigma(X^\sigma)^{-1}$. \(\square\)
3 Reduction of Order Theorems

In this section we establish two related reduction of order theorems; first, we need the following preparatory lemma, which allows us to rewrite a delta integral as a nabla integral.

**Lemma 3.1.** Let \( t_0 \in \mathbb{T}_\kappa \), and assume \( X \) is invertible on \( \mathbb{T}_\kappa \). Then

\[
\int_{t_0}^t (X^{*\rho} P^\rho X)^{-1}(s) \nabla s = \int_{t_0}^t (X^{*\sigma} P^\sigma X)^{-1}(s) \Delta s.
\]

**Proof.** Set

\[
Y(t) := \int_{t_0}^t (X^{*\rho} P^\rho X)^{-1}(s) \nabla s - \int_{t_0}^t (X^{*\sigma} P^\sigma X)^{-1}(s) \Delta s;
\]
clearly \( Y(t_0) = 0 \), and

\[
Y^\Delta(t) = \left[ \int_{t_0}^t (X^{*\rho} P^\rho X)^{-1}(s) \nabla s \right]^\Delta - (X^{*\sigma} P^\sigma X)^{-1}(t).
\]

Using Theorem 7.2 (iii) and the set \( B \) in (7.2),

\[
\left[ \int_{t_0}^t (X^{*\rho} P^\rho X)^{-1}(s) \nabla s \right]^\Delta = \begin{cases} (X^{*\rho} P^\rho X)^{-1}(t) & : t \in \mathbb{T}_\kappa \setminus B, \\ \lim_{s \to t^+} (X^{*\rho} P^\rho X)^{-1}(s) & : t \in B. \end{cases}
\]

For \( t \in \mathbb{T}_\kappa \setminus B \), \( \rho(\sigma(t)) = t \), so that \((X^{*\rho} P^\rho X)^{-1}(\sigma(t)) = (X^{*\sigma} P^\sigma X)^{-1}(t)\). For \( t \in B \), \( t = \sigma(t) \) and \( \lim_{s \to t^+} \rho(s) = t \), yielding

\[
\lim_{s \to t^+} (X^{*\rho} P^\rho X)^{-1}(s) = (X^{*\sigma} P^\sigma X)^{-1}(t) = (X^{*\sigma} P^\sigma X)^{-1}(t).
\]

Thus in either case \( Y^\Delta(t) = 0 \). By the uniqueness property, \( Y \equiv 0 \), and the result follows.

**Theorem 3.2** (Reduction of Order I). Let \( t_0 \in \mathbb{T}_\kappa \), and assume \( X \) is a prepared solution of (2.1) with \( X \) invertible on \( \mathbb{T}_\kappa \). Then a second prepared solution \( Y \) of (2.1) is given by

\[
Y(t) := X(t) \int_{t_0}^t (X^{*\sigma} P^\sigma X)^{-1}(s) \Delta s, \quad t \in \mathbb{T}_\kappa
\]
such that \( X, Y \) are normalized prepared bases of (2.1).

**Proof.** The argument here is an extension of the scalar case; see [17, Theorem 3.9]. For \( Y \) defined above, by the product rule for delta derivatives we have

\[
Y^\Delta = P^{-1}(X^{*\rho})^{-1} + X^\Delta X^{-1} Y,
\]
which is continuous on $T^\kappa$ since $X \in \mathbb{D}$ and $P$ is continuous. For $W$ given in Definition 2.12,
\[
W(X,Y) = X^*PY^\Delta - (PX^\Delta)^*Y
\]
\[
= X^*P(P^{-1}(X^*)^{-1} + X^\Delta X^{-1}Y) - (PX^\Delta)^*Y
\]
\[
= I + X^*PX^\Delta X^{-1}Y - (X^\Delta)^*PY
\]
\[
= I + (X^*PX^\Delta - (X^\Delta)^*PX)X^{-1}Y = I
\]
since $X^*PX^\Delta$ is Hermitian by Theorem 2.22 (ii). Next, we check for left-dense continuity:
\[
(PY^\Delta)^\nabla = ((X^*)^{-1} + PX^\Delta X^{-1}Y)^\nabla
\]
\[
= -(X^*)^{-1}(X^\nabla)^{-1}((X^*)^{-1} + PX^\Delta X^{-1}Y)^\nabla
\]
\[
+ (PX^\Delta)^\rho \int_{t_0}^t (X^*PX^\sigma)^{-1}(s)\Delta s \right)^\nabla.
\]
Clearly the first and second terms on the right-hand side are left-dense continuous, as $(PX^\Delta)^\rho = P^\rho X^\nabla$. By Theorem 7.2 (ii), Theorem 7.4 (i) and Lemma 3.1, we have
\[
\int_{t_0}^t (X^*PX^\sigma)^{-1}(s)\Delta s \right)^\nabla = (X^*P^\rho X)^{-1}(t).
\]
As $P$ and $X$ are continuous and $\rho$ is left-dense continuous, we get that $Y \in \mathbb{D}$.

By Theorem 2.20, $W(X,Y) = I$ guarantees $Y$ is a solution of (2.1). To see that $Y$ is prepared, note that
\[
Y^*PY^\Delta = Y^*P(P^{-1}(X^*)^{-1} + X^\Delta X^{-1}Y) = Y^*(X^*)^{-1} + Y^*(PX^\Delta X^{-1}Y)
\]
\[
= (X^{-1}Y)^* + Y^*ZY = \int_{t_0}^t (X^*PX^\sigma)^{-1}(s)\Delta s + Y^*ZY,
\]
which is Hermitian by Lemma 2.23 since $X$ is prepared and $Z$ is Hermitian. Consequently, $X, Y$ are normalized prepared bases for (2.1).

**Lemma 3.3.** Assume $X, Y \in \mathbb{D}$ are normalized prepared bases of (2.1). Then $U := XE + YF$ is a prepared solution of (2.1) for constant $n \times n$ matrices $E, F$ if and only if $F^*E$ is Hermitian. If $F = I$, then $X, U$ are normalized prepared bases of (2.1) if and only if $E$ is a constant Hermitian matrix.

**Proof.** Assume $X, Y \in \mathbb{D}$ are normalized prepared bases of (2.1). Then by Theorem 2.22 and Definition 2.12,
\[
X^*PX^\Delta = X^\Delta^*PX, \quad Y^*PY^\Delta = Y^\Delta^*PY, \quad X^*PY^\Delta - X^\Delta^*PY = I.
\]
By linearity \( U := XE + YF \) is a solution of (2.1). Checking appropriate Wronskians,

\[
W(U, U) = U^*PU^\Delta - U^\Delta*PU
\]

\[
= (E^*X^* + F^*Y^*)P(X^\Delta E + Y^\Delta F)
\]

\[
- (E^*X^\Delta + F^*Y^\Delta^*)P(XE + YF)
\]

\[
= E^*(X^*PX^\Delta - X^\Delta*PX)E + F^*(Y^*PY^\Delta - Y^\Delta*PY)F
\]

\[
+ E^*(X^*PY^\Delta - X^\Delta*PY)F + F^*(Y^*PX^\Delta - Y^\Delta*PX)E
\]

\[
= 0 + 0 + E^*IF - F^*IE,
\]

and

\[
W(X, U) = X^*PU^\Delta - X^\Delta*PU = X^*P(X^\Delta E + Y^\Delta F) - X^\Delta*P(XE + YF) = F.
\]

Clearly the first claim holds. If \( F = I \), then \( W(X, U) = I \), and \( U = XE + Y \) is a prepared solution of (2.1) if and only if \( E \) is a constant Hermitian matrix.

**Theorem 3.4 (Reduction of Order II).** Let \( t_0 \in \mathbb{T}^* \), and assume \( X \) is a prepared solution of (2.1) with \( X \) invertible on \( \mathbb{T} \). Then \( U \) is a second \( n \times n \) matrix solution of (2.1) iff \( U \) satisfies the first-order matrix equation

\[
(X^{-1}U)^\Delta(t) = (X^*PX^\sigma)^{-1}(t)F, \quad t \in \mathbb{T}^*, \quad t \geq t_0, \quad (3.1)
\]

for some constant \( n \times n \) matrix \( F \) iff \( U \) is of the form

\[
U(t) = X(t)E + X(t) \left( \int_{t_0}^t (X^*PX^\sigma)^{-1}(s) \Delta s \right) F, \quad t \in \mathbb{T}, \quad t \geq t_0, \quad (3.2)
\]

where \( E \) and \( F \) are constant \( n \times n \) matrices. In the latter case,

\[
E = X^{-1}(t_0)U(t_0), \quad F = W(X, U)(t_0), \quad (3.3)
\]

such that \( U \) is a prepared solution of (2.1) iff \( F^*E = E^*F \).

**Proof.** Assume \( X \) is a prepared solution of (2.1) with \( X \) invertible on \( \mathbb{T} \). Let \( U \) be any \( n \times n \) matrix solution of (2.1); we must show \( U \) is of the form (3.2). Using the Wronskian from Definition 2.12, set

\[
F := W(X, U)(t_0) = X^*PU^\Delta - X^\Delta*PU.
\]

Since

\[
(X^{-1}U)^\Delta = -(X^\sigma)^{-1}X^\Delta X^{-1}U + (X^\sigma)^{-1}U^\Delta
\]

and \( X \) is prepared we have that

\[
(X^*PX^\sigma)^{-1}F = (X^\sigma)^{-1}U^\Delta - (X^*PX^\sigma)^{-1}X^\Delta*PU
\]

\[
= (X^{-1}U)^\Delta + (X^\sigma)^{-1}X^\Delta X^{-1}U
\]

\[
- (X^\sigma)^{-1}P^{-1}(X^*)^{-1}X^\Delta*PU
\]

\[
= (X^{-1}U)^\Delta + (X^\sigma)^{-1}P^{-1}PX^\Delta X^{-1}U
\]

\[
- (X^\sigma)^{-1}P^{-1}(PX^\Delta X^{-1})^*U
\]

\[
= (X^{-1}U)^\Delta.
\]
Delta integrating both sides from \( t_0 \) to \( t \),
\[
X^{-1}(t)U(t) - X^{-1}(t_0)U(t_0) = \left( \int_{t_0}^t (X^*PX^*)^{-1}(s) \Delta s \right) F;
\]
recovering \( U \) yields
\[
U(t) = X(t)E + X(t) \left( \int_{t_0}^t (X^*PX^*)^{-1}(s) \Delta s \right) F
\]
provided \( E = X^{-1}(t_0)U(t_0) \).

Conversely, assume \( U \) is given by (3.2). By Theorem 3.2 and linearity \( U \) is a solution of (2.1) on \( \mathbb{T}^\kappa \) for \( t \geq t_0 \). Setting \( t = t_0 \) in (3.2) leads to \( E \) in (3.3). By the constancy of the Wronskian, \( W(X,U)(t) \equiv W(X,U)(t_0) \); suppressing the \( t_0 \), and using (3.2) and the fact that \( X \) is prepared,
\[
W(X,U) = X^*PU^\Delta - X^\Delta*PU = X^*P \left[ X^\Delta E + P^{-1}(X^*)^{-1}F \right] - X^\Delta*PU
\]
\[
= X^*PX^\Delta E + F - X^\Delta*PXE = F.
\]
From Lemma 3.3, \( U \) is a prepared solution of (2.1) iff \( F^*E \) is Hermitian. \( \Box \)

4 Riccati matrix equation

Now we will consider the associated Riccati dynamic matrix equation
\[
RZ = 0, \quad \text{where} \quad RZ := Z^\nabla + Q(t) + (Z^\rho)^* \left\{ P^\rho(t) + \nu(t)Z^\rho \right\}^{-1} Z^\rho. \quad (4.1)
\]

Definition 4.1. Denote by \( \mathbb{D}_R \) the set of all matrix functions \( Z : \mathbb{T}^\kappa \rightarrow \mathbb{R} \) such that \( Z^\nabla : \mathbb{T}^\kappa \rightarrow \mathbb{R} \) is left-dense continuous and such that \( P^\rho(t) + \nu(t)Z^\rho(t) \) is invertible for all \( t \in \mathbb{T}^\kappa \). A matrix \( Z \in \mathbb{D}_R \) is a solution of (4.1) on \( \mathbb{T}^\kappa \) iff \( RZ(t) = 0 \) on \( \mathbb{T}^\kappa \).

Theorem 4.2. If (2.1) has a prepared solution \( X \) with \( X \) invertible on \( \mathbb{T} \), then \( Z \) defined by
\[
Z(t) := P(t)X^\Delta(t)X^{-1}(t)
\]
for \( t \in \mathbb{T}^\kappa \) is a Hermitian solution of (4.1) on \( \mathbb{T}^\kappa \). Conversely, if (4.1) has a Hermitian solution \( Z \) on \( \mathbb{T}^\kappa \), then there exists a prepared solution \( X \) of (2.1) such that \( X \) is invertible on \( \mathbb{T} \) and \( Z(t) = P(t)X^\Delta(t)X^{-1}(t) \).

Proof. First assume (2.1) has a prepared solution \( X \) that is invertible on \( \mathbb{T} \). Let \( Z(t) = P(t)X^\Delta(t)X^{-1}(t) \). Then by Lemma 2.23, \( Z \) is Hermitian for all \( t \in \mathbb{T}^\kappa \). Also we have by Theorem 7.1 and Theorem 7.4 that
\[
P^\rho + \nu Z^\rho = P^\rho + \nu P^\rho X^\Delta(X^{-1})^\rho = P^\rho(X^\rho + \nu X^\nabla)(X^{-1})^\rho = P^\rho X(X^{-1})^\rho,
\]

(4.2)
which is invertible for all $t \in T^\kappa_\rho$. Moreover,

\[
Z^\nabla = (PX^\Delta X^{-1})^\nabla \\
= (PX^\Delta)^\rho(X^{-1})^\nabla + (PX^\Delta)^\nabla X^{-1} \\
= -(PX^\Delta)^\rho X^{-1}X^\nabla(X^\rho)^{-1} + (PX^\Delta)^\nabla X^{-1}
\]

is left-dense continuous on $T^\kappa_\rho$ since $X \in \mathbb{D}$. Thus $Z \in \mathbb{D}_R$. Now we have that

\[
RZ = Z^\nabla + Q + Z^\rho[P^\rho + \nu Z^\rho]^{-1}Z^\rho \\
= (PX^\Delta X^{-1})^\nabla + Q + Z^\rho[P^\rho + \nu P^\rho X^\Delta X^{-1}]^\rho^{-1}Z^\rho \\
= (PX^\Delta)^\rho(X^{-1})^\nabla + (PX^\Delta)^\nabla X^{-1} + Q \\
+ Z^\rho[P^\rho(X^\rho + \nu X^\nabla)(X^{-1})^\rho]^{-1}Z^\rho \\
= -(PX^\Delta)^\rho X^{-1}X^\nabla(X^{-1})^\rho - (QX)X^{-1} + Q \\
+ Z^\rho[P^\rho X(X^{-1})^\rho]^{-1}Z^\rho \\
= P^\rho X^\Delta(X^{-1})^\rho X^\rho X^{-1}(P^{-1})^\rho P^\rho X^\Delta X^{-1} \\
= P^\rho X^\nabla X^{-1}X^\nabla(X^{-1})^\rho - P^\rho X^\nabla X^{-1}X^\nabla(X^{-1})^\rho \\
= 0.
\]

Thus $Z$ is a Hermitian solution of (4.1) on $T^\kappa$.

Conversely, let $Z$ be a Hermitian solution of (4.1) on $T^\kappa$. Then $P^\rho + \nu Z^\rho$ is invertible on $T^\kappa_\rho$ and $Z$ is continuous. Since $P^\rho + \nu Z^\rho$ is invertible on $T^\kappa_\rho$, $(P^{-1})^\rho[P^\rho + \nu Z^\rho] = I + \nu(P^{-1}Z)^\rho$ is invertible on $T^\kappa_\rho$. Hence $-(P^{-1}Z)^\rho$ is $\nu$-regressive and thus $P^{-1}Z$ is regressive by Lemma 2.6. Also we know that $P^{-1}Z$ is right-dense continuous. So let $t_0 \in T$ and put $X = e_{P^{-1}Z}(t, t_0)$. Then $X$ is well defined and invertible. Note also that $X^\Delta = P^{-1}ZX$ is continuous. Furthermore,

\[
(PX^\Delta)^\nabla = (ZX)^\nabla = Z^\nabla X^\rho + ZX^\nabla = Z^\rho X^\nabla + Z^\nabla X^\rho + ZX^\Delta X^\rho
\]

is left-dense continuous on $T^\kappa_\rho$, and

\[
(PX^\Delta)^\nabla = (ZX^\rho)^\nabla \\
= Z^\nabla X^\rho + Z^\rho X^\nabla \\
= Z^\rho X^\Delta X^\rho + (-Q - Z^\rho[P^\rho + \nu Z^\rho]^{-1}Z^\rho)X \\
= -(QX - Z^\rho[P^\rho + \nu Z^\rho]^{-1}Z^\rho X + Z^\rho[P^{-1}ZX]^\rho \\
= -(QX + Z^\rho[P^\rho + \nu Z^\rho]^{-1}(-Z^\rho X + [P^\rho + \nu Z^\rho][P^{-1}ZX]^\rho) \\
= -(QX + Z^\rho[P^\rho + \nu Z^\rho]^{-1}(-Z^\rho X + Z^\rho X^\rho + \nu Z^\rho(P^{-1})^\rho Z^\rho X^\rho) \\
= -(QX + Z^\rho[P^\rho + \nu Z^\rho]^{-1}(-Z^\rho X + Z^\rho(X - \nu X^\Delta X^\rho) \\
+ \nu Z^\rho(P^{-1})^\rho Z^\rho X^\rho) \\
= -(QX + Z^\rho[P^\rho + \nu Z^\rho]^{-1}(-Z^\rho X + Z^\rho X - \nu Z^\rho(P^{-1})^\rho Z^\rho X^\rho \\
+ \nu Z^\rho(P^{-1})^\rho Z^\rho X^\rho) \\
= -QX.
\]
This shows that $X$ is a solution of (2.1) on $T^*_\kappa$. Now note that

$$X^*(t_0) P(t_0) X^\Delta(t_0) = P(t_0) X^\Delta(t_0) = Z(t_0) X(t_0) = Z(t_0),$$

since $X(t_0) = I$; as $Z$ is Hermitian, $X^*(t_0) P(t_0) X^\Delta(t_0)$ is Hermitian, too. By Theorem 2.22, this implies that $X$ is a prepared solution of (4.1). \hfill \Box

**Theorem 4.3.** The self-adjoint matrix equation (2.1) has a prepared solution $X$ on $T$ with $X^\rho P^\rho X > 0$ on $T^*_\kappa$ if and only if the Riccati equation equation (4.1) has a Hermitian solution $Z$ on $T^\kappa$ satisfying $P^\rho + \nu Z^\rho > 0$ on $T^\kappa$.

**Proof.** Assuming the Riccati substitution $Z = P X^\Delta X^{-1}$, recall from (4.2) that $P^\rho + \nu Z^\rho = P^\rho X (X^{-1})^\rho$, which leads to

$$(X^\rho)^*(P^\rho + \nu Z^\rho)X^\rho = (X^\rho)^*P^\rho X (X^{-1})^\rho X^\rho = (X^\rho)^*P^\rho X. \quad (4.3)$$

Thus if (2.1) has a prepared solution $X$ on $T$ with $X^\rho P^\rho X > 0$ on $T^*_\kappa$, then $X$ is invertible. Setting $Z = PX^\Delta X^{-1}$, by Theorem 4.2 we know that $Z$ is a Hermitian solution of (4.1) on $T^\kappa$. From (4.3) we have

$$X^\rho P^\rho X > 0 \text{ on } T^*_\kappa \implies X^\rho [P^\rho + \nu Z^\rho] X^\rho > 0 \text{ on } T^*_\kappa \implies P^\rho + \nu Z^\rho > 0 \text{ on } T^*_\kappa.$$

Now suppose (4.1) has Hermitian solution $Z$ on $T^\kappa$ satisfying $P^\rho + \nu Z^\rho > 0$ on $T^*_\kappa$. Then Theorem 4.2 says that there exists an invertible prepared solution $X$ of (2.1) with $Z = PX^\Delta X^{-1}$. Once again using (4.3) we have

$$P^\rho + \nu Z^\rho > 0 \text{ on } T^*_\kappa \implies X^\rho [P^\rho + \nu Z^\rho] X^\rho > 0 \text{ on } T^*_\kappa \implies X^\rho P^\rho X > 0 \text{ on } T^*_\kappa,$$

and the proof is complete. \hfill \Box

Another way to prove the above theorem is to use the following Factorization Theorem.

**Theorem 4.4 (Factorization Theorem).** Assume $X \in D$ satisfies $X^\rho P^\rho X > 0$ on $T^*_\kappa$ and $Z$ is defined by the Riccati substitution

$$Z(t) = P(t) X^\Delta(t) X^{-1}(t)$$

for $t \in T^\kappa$. Then $Z \in D_R$ and $LX(t) = RZ(t) X(t)$ for $t \in T^*_\kappa$.

**Proof.** We first wish to show that $Z \in D_R$. Note that

$$Z^\nabla(t) = \left[ P(t) X^\Delta(t) X^{-1}(t) \right]^\nabla$$

$$= [P(t) X^\Delta(t)]^\rho (X^{-1}(t))^\nabla + [P(t) X^\Delta(t)]^\nabla (X^{-1}(t))^\rho$$

$$= -[P(t) X^\Delta(t)]^\rho X^{-1}(t) X^\nabla(t) (X^\rho(t))^{-1} + [P(t) X^\Delta(t)]^\nabla X^{-1}(t),$$

which is left-dense continuous on $T^*_\kappa$ since $X \in D$. From (4.2)

$$P^\rho(t) + \nu(t) Z^\rho(t) = P^\rho(t) X(t) X^{-1}(t)^\rho(t)$$
is invertible for all $t \in \mathbb{T}^*_k$ since $X^{*p}P^pX > 0$ on $\mathbb{T}_k$ implies that $X$ is nonsingular on $\mathbb{T}_k$. Thus $Z \in \mathbb{D}_R$. It remains to show that $LX(t) = RZ(t)X(t)$ for $t \in \mathbb{T}^*_k$.

Suppressing the arguments, we get

$$RZX = (Z^\nabla + Q + Z^\rho [P^\rho + \nu Z^\rho]^{-1} Z^\rho)X$$

$$= ((PX^\Delta X^{-1})^\nabla + Q + Z^\rho [P^\rho X X^{-1}^p]^{-1} Z^\rho)X$$

$$= (-P^p X^\Delta^p X^{-1} X^\nabla X^{p^{-1}} + [PX^\Delta]X^{-1} + Q$$

$$+ P^p X^\Delta^p X^{-1} X^\nabla X^{p^{-1}} P^p X^\Delta^p X^{-1}^p)X$$

$$= (-P^p X^\nabla X^{-1} X^\nabla X^{p^{-1}} + [PX^\Delta]X^{-1} + Q$$

$$+ P^p X^\nabla X^{-1} X^\nabla X^{p^{-1}})X$$

$$= [PX^\Delta]X + QX$$

$$= LX.$$

Hence the theorem is proved. 

\[\square\]

5 Quadratic Functional and Picone Identity

We are now interested in exploring the connection between the disconjugacy of the vector form of (2.1),

$$Lx(t) = (Px^\Delta)^\nabla (t) + Q(t)x(t) = 0,$$  \hspace{1cm} (5.1)

and the positive definiteness of a certain related quadratic functional. Note that a prepared solution $x$ of (5.1) satisfies $W(x,x) = 0$ for the generalized Wronskian given in Definition 2.12. To this end, for $a,b \in \mathbb{T}_k$, we define the set of admissible functions $A$ to be

$$A := \{ \eta \in C^1_{pld}([\rho(a), \sigma(b)]_\tau, \mathbb{R}^n) \setminus \{0\} : \eta(\rho(a)) = \eta(\sigma(b)) = 0 \} .$$

Here $C^1_{pld}([\rho(a), \sigma(b)]_\tau, \mathbb{R}^n)$ denotes the set of all continuous functions whose nabla derivatives are piece-wise left-dense continuous. Then we define the quadratic functional $\mathcal{F}$ on $A$ via

$$\mathcal{F}(\eta) := \int_{\rho(a)}^{\sigma(b)} \left[ \eta^\nabla^* P^\rho \eta^\nabla - \eta^* Q \eta \right] dt.$$  \hspace{1cm} (5.2)

**Definition 5.1.** The functional $\mathcal{F}$ is positive definite on $A$ (write $\mathcal{F} > 0$) provided $\mathcal{F}(\eta) \geq 0$ for all $\eta \in A$, and $\mathcal{F}(\eta) = 0$ iff $\eta = 0$.

**Lemma 5.2.** Let $u$ be a prepared vector solution of (5.1) on $[\rho(a), \sigma(b)]_\tau$, and let $c, d \in [\rho(a), \sigma(b)]_\tau$ with $\rho(a) \leq c < \rho(d) \leq d \leq \sigma(b)$. If $c = \rho(a)$ or $c$ is a left-dense right-dense point, assume $u(c) = 0$; if $d$ is a left-dense right-dense point, assume $u(d) = 0$. Then for

$$\eta(t) := \begin{cases} u(t) & \text{if } t \in (\rho(c), \rho(d)]_\tau \\ 0 & \text{otherwise}, \end{cases}$$
$$\eta \in \mathbb{A} \text{ and } \mathcal{F}(\eta) = k + r,$$

where

$$k = \begin{cases} 
- (u^* P u^\Delta) (c) : \nu(c) = 0, \\
\frac{1}{\nu(c)} (u^* P^\rho u) (c) : \nu(c) > 0,
\end{cases} \quad \text{and} \quad r = \begin{cases} 
(u^* P u^\Delta) (d) : \nu(d) = 0, \\
\frac{1}{\nu(d)} (u^* P^\rho u) (d) : \nu(d) > 0.
\end{cases}$$

**Proof.** Let $u, \eta$ be as described in the statement of the lemma. It is apparent that $\eta \in C_{p.d}^1([\rho(a), \sigma(b)]_\tau, \mathbb{R}^n)$ with $\eta(\rho(a)) = 0$ and $\eta(\sigma(b)) = 0$. Thus, $\eta \in \mathbb{A}$. Now consider $\mathcal{F}(\eta)$ for $\mathcal{F}$ in (5.2). We have $\eta = 0 = \nabla^* \text{ on } [\rho(a), \rho(c)]_\tau \cup (\rho(d), \sigma(b)]_\tau$, so

$$\mathcal{F}(\eta) = \int_{\rho(c)}^{\rho(d)} \left[ \eta \nabla^* P^\rho \eta \nabla - \eta^* Q \eta \right] (t) \nabla t$$

since $\eta(\rho(d)) = 0$ or $\nu(\rho) = 0$ zeros out $\nu(\rho) (\eta^* Q \eta)(d)$. Pulling out the $\rho$ and using integration by parts,

$$\int_{\rho(c)}^{\rho(d)} \left[ u \nabla^* (P u^\Delta)^\rho \right] (t) \nabla t = u^* P u^\Delta \bigg|_{\rho(c)}^{\rho(d)} - \int_{\rho(c)}^{\rho(d)} \left[ u \nabla^* (P u^\Delta) \nabla \right] (t) \nabla t,$$

yielding

$$\mathcal{F}(\eta) = \nu(c) (\eta \nabla^* P^\rho \eta \nabla)(c) - \nu(c) (\eta^* Q \eta)(c) + \nu(d) (\eta \nabla^* P^\rho \eta \nabla)(d) + u^* P u^\Delta \bigg|_{\rho(c)}^{\rho(d)}$$

since $u$ solves $Lu = 0$. Let

$$k = \nu(c) (\eta \nabla^* P^\rho \eta \nabla)(c) - \nu(c) (\eta^* Q \eta)(c) - (u^* P u^\Delta)(c)$$

and

$$r = \nu(d) (\eta \nabla^* P^\rho \eta \nabla)(d) + (u^* P u^\Delta)^\rho(d).$$

If $\nu(c) = 0$ or $\nu(d) = 0$ the lemma holds. If $\nu(c) > 0$, then $c$ is a left-scattered point and, suppressing the $c$, we have

$$k = \frac{1}{\nu} (u^* P^\rho u) - \nu (u^* Qu) - (u^* P u^\Delta) + (u^* P^\rho u^\nabla) - (u^* P^\rho u^\nabla)$$

$$= \frac{1}{\nu} (u^* P^\rho u) - \frac{1}{\nu} (u^* P^\rho u^\nabla) - \nu u^* \left( Qu + \frac{P u^\Delta}{\nu} - \frac{(P u^\Delta)^\rho}{\nu} \right)$$

$$= \frac{1}{\nu} (u^* P^\rho u - u^* P^\rho (u - u^\rho)) - \nu u^* (Qu + (P u^\Delta)^\nabla)$$

$$= \frac{1}{\nu} (u^* P^\rho u^\nabla) = \frac{1}{\nu} (u^* P^\rho u)$$
as $u$ is a prepared vector solution of (2.1). In a similar manner, if $\nu(d) > 0$, then $d$ is a left-scattered point and, suppressing the $d$, we have

$$r = \nu(d)(\eta\nabla^{*} P^{\rho} \eta \nabla)(d) + (u^{*} Pu \Delta)^{\rho}(d) = \frac{1}{\nu}(u^{*} Pu \Delta) + (u^{*} P \eta \nabla)$$

$$= \frac{1}{\nu} u^{*} P \eta \nabla (u^\nu + \nu u \nabla) = \frac{1}{\nu} (u^{*} P \eta \nabla)$$

using Theorem 7.1. The proof is complete.

**Theorem 5.3** (Picone’s Identity). Assume the matrix Riccati equation (4.1) has a Hermitian solution $Z$ on $[\rho(a), b]_{\tau}$ satisfying

$$P^{\rho}(t) + \nu(t)Z^{\rho}(t) > 0 \quad \text{for all} \quad t \in [a, \sigma(b)]_{\tau}.$$  

Then the self-adjoint matrix equation (2.1) has on $[\rho(a), \sigma(b)]_{\tau}$ a prepared invertible solution $X$ with $X^{*} P^{\rho} X > 0$ on $[a, \sigma(b)]_{\tau}$, and we can take

$$D := X^{\rho} X^{-1}(P^{\rho})^{-1}$$

on $[a, \sigma(b)]_{\tau}$. Then $D^{-1} > 0$ on $[a, \sigma(b)]_{\tau}$, and for any $\eta \in \mathbb{R}$, on $t \in [a, \sigma(b)]_{\tau}$ we have the equality

$$(\eta^{*}Z\eta)^{\nabla}(t) = \eta^{\nabla*}(t)P^{\rho}(t)\eta(t) - \eta^{*}(t)Q(t)\eta(t) - (\eta^{*} - DZ^{\rho}\eta)^{*} D^{-1} (\eta^{*} - DZ^{\rho}\eta).$$

**Proof.** The first claim follows from the assumptions and Theorem 4.3. Using (4.2),

$$P^{\rho} + \nu Z^{\rho} = P^{\rho}X(X^{*})^{-1} \Rightarrow X^{*} P^{\rho} + \nu Z^{\rho} X = X^{*} P^{\rho} X$$

implies

$$P^{\rho} + \nu Z^{\rho} > 0 \quad \iff \quad X^{*} P^{\rho} X > 0$$

and

$$D^{-1}(t) = (P^{\rho}X(X^{*})^{-1})(t) = (P^{\rho} + \nu Z^{\rho})(t) > 0, \quad t \in [a, \sigma(b)]_{\tau}.$$  

For any $\eta \in \mathbb{R}$, on $[a, \sigma(b)]_{\tau}$ we have

$$(\eta^{*}Z\eta)^{\nabla} = \eta^{\nabla*}(Z\eta)^{\rho} + \eta^{*}(Z\eta)^{\nabla}$$

$$= \eta^{\nabla*} Z^{\rho} \eta^{\nabla} + \eta^{*} Z^{\rho} \eta^{\nabla} + \eta^{*}(-Q - Z^{\rho}(P^{\rho} + \nu Z^{\rho})^{-1} Z^{\rho}) \eta$$

$$= \eta^{\nabla*} Z^{\rho} \eta - \eta^{*} \eta^{\nabla} + \eta^{*} Z^{\rho} \eta - \eta^{*} Z^{\rho} DZ^{\rho} \eta$$

$$= \eta^{\nabla*} Z^{\rho} \eta - \eta^{*} \nu Z^{\rho} \eta^{\nabla} + \eta^{*} Z^{\rho} \eta^{\nabla} - \eta^{*} Q \eta - \eta^{*} Z^{\rho} DZ^{\rho} \eta$$

$$+ \eta^{*} P \eta^{\nabla*} - \eta^{*} \nu P \eta^{\nabla}$$

$$= \eta^{\nabla*} P \eta^{\nabla} - \eta^{*} Q \eta + \eta^{\nabla*} Z^{\rho} \eta - \eta^{*} D^{-1} \eta^{\nabla} + \eta^{*} Z^{\rho} \eta^{\nabla} - \eta^{*} Z^{\rho} DZ^{\rho} \eta$$

$$= \eta^{\nabla*} P \eta^{\nabla} - \eta^{*} Q \eta - (\eta^{\nabla} - DZ^{\rho} \eta)^{*} D^{-1} (\eta^{\nabla} - DZ^{\rho} \eta),$$

since $Z$ and $D$ are Hermitian.
6 Reid Roundabout Theorem

Our previous work culminates here with the following roundabout theorem which generalizes, consolidates, streamlines and extends results found in the continuous case by Reid [18, Theorem 2.1], the discrete case by Bohner and Peterson [4, Theorem 5.13], the time-scale self-adjoint scalar case by Messer [11, Theorem 4.68], and the delta delta matrix case by Bohner and Peterson [10, Section 5.3]. First, we need the next definition.

**Definition 6.1.** A prepared vector solution $x$ of (5.1) has a generalized zero at $\rho(a)$ iff $x^\rho(a) = 0$, and $x$ has a generalized zero at $t_0 > \rho(a)$ iff $x(t_0) = 0$, or if $t_0$ is a left-scattered point with

$$x^\rho(t_0) \neq 0 \quad \text{and} \quad (x^*P^\rho x)(t_0) < 0.$$ 

The vector equation (5.1) is disconjugate on $[\rho(a),\sigma(t)]$ iff no nontrivial prepared solution has two (or more) generalized zeros in $[\rho(a),\sigma(t)]$.

**Theorem 6.2** (Reid Roundabout Theorem). Let $P$ and $Q$ be Hermitian $n \times n$-matrix-valued functions on a time scale $\mathbb{T}$ such that $P > 0$ and $Q$ are continuous for all $t \in \mathbb{T}$. Then the following are equivalent:

(i) There is a prepared solution $X$ of (2.1) with $X^*P^\rho X > 0$ on $[a,\sigma(b)]_\mathbb{T}$.

(ii) The Riccati equation (4.1) has a Hermitian solution $Z$ on $[\rho(a),b]_\mathbb{T}$ with $P^\rho + \nu Z^\rho > 0$ on $[a,\sigma(b)]_\mathbb{T}$.

(iii) $F$ is positive definite on $\mathbb{K}$, for $F$ given in (5.2).

(iv) The self-adjoint vector equation (5.1) is disconjugate on $[\rho(a),\sigma(b)]_\mathbb{T}$.

(v) Principal solution $U$ of (2.1) at $\rho(a)$ satisfies $U^*P^\rho U > 0$ on $[a,\sigma(b)]_\mathbb{T}$.

(vi) The unique solution $V$ of the initial value problem

$$LV = 0, \quad V(b) = -\mu(b)P^{-1}(b), \quad V^\Delta(b) = P^{-1}(\sigma(b))$$

satisfies $V^*P^\rho V > 0$ on $[a,\sigma(b)]_\mathbb{T}$.

**Proof.** Note that $(i) \iff (ii)$ by Theorem 4.3.

$(ii) \iff (iii)$: From Picone’s identity, Theorem 5.3, there exists a prepared solution $X$ with $X$ invertible on $[\rho(a),\sigma(b)]_\mathbb{T}$ with $(X^*P^\rho X) > 0$ on $[a,\sigma(b)]_\mathbb{T}$. Let $Z = PX^\Delta X^{-1}$; by Theorem 4.3, $Z$ is a Hermitian solution of (4.1) on $[\rho(a),b]_\mathbb{T}$ satisfying $P^\rho + \nu Z^\rho > 0$ on $[a,\sigma(b)]_\mathbb{T}$. By Picone’s identity, Theorem 5.3, for any $\eta \in \mathbb{K}$ on $[a,\sigma(b)]_\mathbb{T}$ and $D := PX^{-1}(P^\rho)^{-1}$ we have

$$\langle \eta^*Z\eta \rangle^\Delta (t) = \eta^*X(t)P^\rho(t)\eta^\Delta (t) - \eta^*U(t)Q(t)\eta(t) - (\eta^*DZ^\rho\eta)^*D^{-1}(\eta^*DZ^\rho\eta).$$
As the equation holds on \([a, \sigma(b)]_{\mathbb{T}}\), we may nabla integrate from \(\rho(a)\) to \(\sigma(b)\) to obtain
\[
\mathcal{F}(\eta) = \int_{\rho(a)}^{\sigma(b)} (\eta^\nabla - D Z^\rho \eta)^* (t) D^{-1}(t) (\eta^\nabla - D Z^\rho \eta) (t) \nabla t
\]
since \(\eta\) is admissible. As \(D^{-1} > 0\), we have \(\mathcal{F}(\eta) \geq 0\) for all \(\eta \in \mathbb{A}\). Furthermore, if \(\eta \equiv 0\), then \(\mathcal{F}(\eta) = 0\). If \(\mathcal{F}(\eta) = 0\) for some \(\eta \in \mathbb{A}\), then on \([a, \sigma(b)]_{\mathbb{T}}\) we have
\[
\eta^\nabla = D Z^\rho \eta = [X^\rho X^{-1} X^\nabla (X^\rho)^{-1}] \eta.
\]
Checking for \(\nu\)-regressivity (see Definition 2.5),
\[
I - \nu X^\rho X^{-1} X^\nabla (X^\rho)^{-1} = X^\rho X^{-1} [X - \nu X^\nabla] (X^\rho)^{-1} = X^\rho X^{-1},
\]
which is invertible on \([a, \sigma(b)]_{\mathbb{T}}\). Thus \(D Z^\rho \in \mathbb{R}_{\nu}\), ensuring by variation of constants [11, Theorem 3.92] that the initial value problem
\[
\eta^\nabla = (D Z^\rho) \eta, \quad \eta(\sigma(b)) = 0
\]
has a unique solution on \([a, \sigma(b)]_{\mathbb{T}}\), namely \(\eta \equiv 0\) on \([a, \sigma(b)]_{\mathbb{T}}\). But \(\eta \in \mathbb{A}\), putting \(\eta(\rho(a)) = 0\) as well. Consequently \(\mathcal{F}\) is positive definite on \(\mathbb{A}\).

(iii) \(\Rightarrow\) (iv): Assume \((5.1)\) is not disconjugate on \([\rho(a), \sigma(b)]_{\mathbb{T}}\). Then there exists a nontrivial prepared solution \(u\) of \((5.1)\) with (at least) two generalized zeros in \([\rho(a), \sigma(b)]_{\mathbb{T}}\); in other words, there exist points \(c, d \in [\rho(a), \sigma(b)]_{\mathbb{T}}\) with \(\rho(a) \leq c < \rho(d) \leq d \leq \sigma(b)\) such that \(u\) has a generalized zero at \(c\) and \(d\).

If \(c = \rho(a)\), then \(u(c) = 0\), and \(u(d) = 0\) or \(d\) is a left-scattered point and \((u^* P^\rho u)(d) \leq 0\). Define
\[
\eta(t) := \begin{cases} 
  u(t) & : t \in (\rho(c), \rho(d)]_{\mathbb{T}} \\
  0 & : \text{otherwise}.
\end{cases}
\]
By Lemma 5.2, \(\eta \in \mathbb{A}\) and \(\mathcal{F}(\eta) \leq 0\), but \(\eta \neq 0\), making \(\mathcal{F}\) not positive definite. By the contrapositive argument, the assertion holds.

(iv) \(\Rightarrow\) (v): Recall from Definition 2.8 that \(U\) is the principal solution of \((2.1)\) at \(\rho(a)\) iff \(U\) satisfies
\[
LU = 0, \quad U(\rho(a)) = 0, \quad U^\Delta(\rho(a)) = P^{-1}(\rho(a));
\]
note that \(U\) is prepared via the initial conditions. Now assume the vector equation \((5.1)\) is disconjugate on \([\rho(a), \sigma(b)]_{\mathbb{T}}\). Fix an arbitrary, constant \(\alpha \in \mathbb{R}^n\), \(\alpha \neq 0\), and set \(u := U \alpha\). Then \(u\) is a nontrivial prepared vector solution of \((5.1)\), with \(u(\rho(a)) = 0\). By disconjugacy, \(u^* P^\rho u > 0\) on \([a, \sigma(b)]_{\mathbb{T}}\), yielding \(\alpha^* U^* P^\rho U \alpha > 0\) on \([a, \sigma(b)]_{\mathbb{T}}\) as well. Yet \(\alpha\) was arbitrary, so \(U^* P^\rho U > 0\) on \([a, \sigma(b)]_{\mathbb{T}}\).

(iv) \(\Rightarrow\) (vi): Assume the vector equation \((5.1)\) is disconjugate on \([\rho(a), \sigma(b)]_{\mathbb{T}}\). The initial value problem \(LV = 0, V(b) = -\mu(b) P^{-1}(b), V^\Delta(b) = P^{-1}[\rho(\sigma(b))]\) has a unique solution \(V\) by Theorem 2.7 which exists on \(\mathbb{T}\). Note that \(V(\sigma(b)) = \ldots\)
0 whether \(b\) is right-scattered or right-dense by Theorem 7.1, ensuring that \(V\) is a prepared solution of (2.1); the proof is thus similar to that of the above case (iv) \(\iff\) (v) and is omitted.

(v) \(\iff\) (vi): Let \(U\) and \(V\) be the unique solutions of (2.1) as described in statements (v) and (vi), respectively. Then \(U\) and \(V\) are prepared solutions with constant Wronskian, and for the set \(B\) in (7.2),

\[
W(U, V) \equiv W(U, V)(\rho(a)) = -V(\rho(a))
\]

\[
= \begin{cases} 
U^*(\sigma(b)) & : b \in \mathbb{T} \setminus B \\
U^*(b)P(b)P^{-1}(\rho(b)) & : b \in B 
\end{cases} = W(U, V)(b). 
(6.1)
\]

If (v) holds, \(U^*(b)\) and \(U^*(\sigma(b))\) are invertible, making \(V(\rho(a))\) invertible also by (6.1). For \(\tau \in (\rho(a), \sigma(b))_T\), \(U\) invertible on \([\tau, \sigma(b)]_T\) implies (i) holds on \([\tau, \sigma(b)]_T\), and thus (ii), (iii), and (iv) likewise hold, as shown above. Consequently, \(V\) has no second generalized zero on \([\tau, \sigma(b)]_T\). By the arbitrary choice of \(\tau\) and the condition of \(V\) at \(\rho(a)\), (vi) holds overall. The fact that (vi) implies (v) follows from a similar argument and is omitted.

(vi) \(\iff\) (i): As we are assuming (vi), condition (v) holds as well. Using (6.1), let

\[
Y(t) := V(t)V^{-1}(\rho(a)) = \begin{cases} 
-\nu V(t)(U^*)^{-1}(\sigma(b)) & : b \in \mathbb{T} \setminus B \\
-\nu V(t)P(b)P^{-1}(b)U^*(b) & : b \in B 
\end{cases}.
\]

Then \(Y\) is a prepared solution of (2.1) with \(W(U, Y) \equiv W(U, Y)(\rho(a)) = -I\), and

\[
Y^*P^0Y > 0 \quad \text{on} \quad [a, \sigma(b)]_T
\]

follows from the assumptions on \(V\). Set \(X := U + Y\); then \(X\) is a prepared solution of (2.1). We must check that \(X^*P^0X > 0\) on \([a, \sigma(b)]_T\). At \(\sigma(b)\),

\[
(X^*P^0X)(\sigma(b)) = [(X^*P^0 + Y^*P^0)U^0](\sigma(b)) = (U^*P^0U)(\sigma(b)) + (Y^*P^0U)(\sigma(b));
\]

the first term here is positive definite, leading us to examine the second. If \(b \in \mathbb{T} \setminus B\), then

\[
(Y^*P^0U)(\sigma(b)) = (Y^*PU^*)(b) = \mu(b)I.
\]

On the other hand, if \(b \in B\), then \(b\) is left-scattered right-dense, \(Y(b) = 0\), and

\[
(Y^*P^0U)(\sigma(b)) = [U^*P^0Y^*]^{\gamma^*(b)} = (U^0 + \nu U^0Y^*P^0Y^*)^{\gamma^*(b)}
\]

\[
= \nu(b) \left[ \frac{1}{\nu} U^*P^0P^0Y^* + U^{\gamma^*}P^0Y^* \right]^{\gamma^*(b)}
\]

\[
= \nu(b) \left[ -U^*P^0Y^* + U^{\gamma^*}P^0Y^* \right]^{\gamma^*(b)}
\]

\[
= \nu(b) \left[ -U^*PY^* + U^{\gamma^*}P^0Y \right]^{\gamma^*(b)}
\]

\[
= -\nu(b)W^0(U, Y)(b) = \nu(b)I.
\]
Consequently, we have for any $b \in \mathbb{T}_\kappa$ that
\[
(X^{*\rho}P^{\rho}X)((\sigma(b)) = \begin{cases}
(U^{*\rho}P^{\rho}U)(\sigma(b)) + \mu(b)I & : b \in \mathbb{T} \setminus B, \\
(U^{*\rho}P^{\rho}U)(\sigma(b)) + \nu(b)I & : b \in B,
\end{cases}
\]
meaning $(X^{*\rho}P^{\rho}X)((\sigma(b)) > 0$.

Finally, let $\tau \in [\rho(a), \sigma(b))_{\mathbb{T}}$. Since $Y$ is a prepared, invertible solution of (2.1) on $[\rho(a), \tau]_{\mathbb{T}}$, by the second reduction of order theorem, Theorem 3.4, and the definition of $Y$ in terms of $V$ we have on $[\rho(a), \tau]_{\mathbb{T}}$ that
\[
X(t) = V(t)V^{-1}(\rho(a)) + V(t) \left( \int_{\rho(a)}^{t} (V^{*\rho}PV^{-1}(s)\nabla s) \right) V^{*}(\rho(a)).
\]

Then
\[
(X^{*\rho}P^{\rho}X)(\tau) = \left[ I + V(\rho(a)) \left( \int_{\rho(a)}^{\rho(\tau)} (V^{*\rho}PV^{-1}(s)\nabla s) \right) V^{*}(\rho(a)) \right]^{*} 
\times V^{-1}(\rho(a))(V^{*\rho}PV^{\rho}(\tau)V^{-1}(\rho(a)) 
\times \left[ I + V(\rho(a)) \left( \int_{\rho(a)}^{\rho(\tau)} (V^{*\rho}PV^{-1}(s)\nabla s) \right) V^{*}(\rho(a)) \right]^{*} 
+ \nu(\tau) \left[ I + V(\rho(a)) \left( \int_{\rho(a)}^{\rho(\tau)} (V^{*\rho}PV^{-1}(s)\nabla s) \right) V^{*}(\rho(a)) \right]^{*}
\geq 0,
\]
where we have used Lemma 3.1 and the fact that $\int_{\rho(t)}^{t} f(s)\nabla s = \nu(t)f(t)$. Given that $\tau \in [\rho(a), \sigma(b))_{\mathbb{T}}$ was arbitrary, and the earlier remark about $X$ at $\sigma(b)$, we have that
\[
(X^{*\rho}P^{\rho}X)(\tau) > 0, \quad \tau \in [\rho(a), \sigma(b))_{\mathbb{T}}
\]
and the proof is complete.

Although the proof of the above roundabout theorem borrows ideas from the proofs of the continuous case [18] and the discrete case [4], it is essentially new for all cases. For example, both [18] and [4] use the Legendre-Clebsch transformation to prove certain implications, but we do not use that transformation here. We do employ the second reduction of order theorem in a key way, however, which they do not. The scalar dynamic equation in [11, Section 4.4] has nothing equivalent to $(v)$ and $(vi)$, and the delta delta matrix case [10, Section 5.3] has individual theorems such as $(i)$ iff $(ii)$, Jacobi’s condition ($(iii)$ iff $(iv)$), the Sturm Separation Theorem ($(v)$ implies $(iv)$) and $(v)$ implies $(iii)$, but no overall unifying roundabout theorem.
7 Overview of Time Scales

Any arbitrary nonempty closed subset of the reals \( \mathbb{R} \) can serve as a time scale \( \mathbb{T} \); see [7, 10, 11, 14]. Here and in the sequel we assume a working knowledge of basic time-scale notation and the time-scale calculus. For more on time-scale definitions and notation, see the special issue [8]. In addition, the following results will prove to be useful.

**Theorem 7.1.** If \( f \) is delta differentiable at \( t \in \mathbb{T}^\kappa \), then \( f^\sigma(t) = f(t) + \mu(t)f^\Delta(t) \). If \( f \) is nabla differentiable at \( t \in \mathbb{T}^\kappa \), then \( f^\rho(t) = f(t) - \nu(t)f^\nabla(t) \).

**Theorem 7.2.** Let \( f : \mathbb{T} \times \mathbb{T} \to \mathbb{R} \) be a continuous function of two variables \((t, s) \in \mathbb{T} \times \mathbb{T}\), and \( a \in \mathbb{T} \). Assume that \( f \) has continuous derivatives \( f^\Delta \) and \( f^\nabla \) with respect to \( t \). Then the following formulas hold:

\[
\begin{align*}
(i) & \quad \left( \int_a^t f(t, s) \Delta s \right)^\Delta = f(\sigma(t), t) + \int_a^t f^\Delta(t, s) \Delta s, \\
(ii) & \quad \left( \int_a^t f(t, s) \Delta s \right)^\nabla = f(\rho(t), \rho(t)) + \int_a^t f^\nabla(t, s) \Delta s, \\
(iii) & \quad \left( \int_a^t f(t, s) \nabla s \right)^\Delta = f(\sigma(t), \sigma(t)) + \int_a^t f^\Delta(t, s) \nabla s, \\
(iv) & \quad \left( \int_a^t f(t, s) \nabla s \right)^\nabla = f(\rho(t), t) + \int_a^t f^\nabla(t, s) \nabla s.
\end{align*}
\]

The following sets and statement [17, Theorem 2.6] (see also [6]) will play an important role in many of our calculations.

**Definition 7.3.** Let the time-scale sets \( A \) and \( B \) be given by

\[
A := \{ t \in \mathbb{T} : t \text{ is a left-dense and right-scattered point} \},
\]

(7.1)

and

\[
B := \{ t \in \mathbb{T} : t \text{ is a right-dense and left-scattered point} \}.
\]

(7.2)

It follows that for \( t \in A \),

\[
\lim_{s \to t^-} \sigma(s) = t,
\]

and for \( t \in \mathbb{T} \setminus A \), \( \sigma(\rho(t)) = t \). Likewise for \( t \in B \),

\[
\lim_{s \to t^+} \rho(s) = t,
\]

and for \( t \in \mathbb{T} \setminus B \), \( \rho(\sigma(t)) = t \).

**Theorem 7.4.** Let the sets \( A \) and \( B \) be given as in (7.1) and (7.2), respectively.

(i) If \( f : \mathbb{T} \to \mathbb{R} \) is \( \Delta \) differentiable on \( \mathbb{T}^\kappa \) and \( f^\Delta \) is right-dense continuous on \( \mathbb{T}^\kappa \), then \( f \) is \( \nabla \) differentiable on \( \mathbb{T}^\kappa \), and

\[
f^\nabla(t) = \begin{cases} f^\Delta(\rho(t)) : t \in \mathbb{T} \setminus A, \\
\lim_{s \to t^+} f^\Delta(s) : t \in A. \end{cases}
\]
(ii) If $f : T \to \mathbb{R}$ is $\nabla$ differentiable on $T_{\kappa}$ and $f^{\nabla}$ is left-dense continuous on $T_{\kappa}$, then $f$ is $\Delta$ differentiable on $T_{\sigma}$, and

$$f^{\Delta}(t) = \begin{cases} f^{\nabla}(\sigma(t)) : t \in T \setminus B, \\ \lim_{s \to t^+} f^{\nabla}(s) : t \in B. \end{cases}$$

The statements of the previous theorem can be formulated as

$$(f^{\Delta})^{\rho} = f^{\nabla}$$ and $$(f^{\nabla})^{\sigma} = f^{\Delta}$$

provided that $f^{\Delta}$ and $f^{\nabla}$ are continuous, respectively.

References


