EXISTENCE OF SOLUTIONS FOR NONLINEAR MULTI-POINT PROBLEMS ON TIME SCALES

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Dedicated to John Baxley on the occasion of his retirement from Wake Forest University.

ABSTRACT. Consider the nonlinear second-order Sturm-Liouville-type multipoint boundary value problem

\[(py^{-})^{\Delta} (t) - q(t)y(t) + h(t)f(t, y(t)) = 0, \quad t_1 < t < t_n,\]
\[\alpha y(t_1) - \beta p(t_1)y^{-}(t_1) = \sum_{i=2}^{n-1} a_i y(t_i), \quad \gamma y(t_n) + \delta p(t_n)y^{-}(t_n) = \sum_{i=2}^{n-1} b_i y(t_i).\]

Conditions for the existence of at least one, two, or three positive solutions on time scales are discussed, employing various fixed point theorems on cones.

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1. INTRODUCTION

We are interested in the second-order multipoint time scale eigenvalue problem

(1.1) \[(py^{-})^{\Delta} (t) - q(t)y(t) + h(t)f(t, y(t)) = 0, \quad t_1 < t < t_n;\]
(1.2) \[\alpha y(t_1) - \beta p(t_1)y^{-}(t_1) = \sum_{i=2}^{n-1} a_i y(t_i), \quad \gamma y(t_n) + \delta p(t_n)y^{-}(t_n) = \sum_{i=2}^{n-1} b_i y(t_i),\]

where

(1.3) \[p, q : [t_1, t_n] \to (0, \infty), \quad p \in C^{\Delta}[t_1, t_n], \quad q \in C[t_1, t_n];\]

the points \(t_i \in \mathbb{T}_\kappa^\infty\) for \(i \in \{1, 2, \ldots, n\}\) with \(t_1 < t_2 < \cdots < t_n;\)

(1.4) \[\alpha, \beta, \gamma, \delta \in [0, \infty), \quad \alpha \gamma + \alpha \delta + \beta \gamma > 0, \quad a_i, b_i \in [0, \infty), \quad i \in \{2, \ldots, n-1\};\]

the continuous function \(f : [0, \infty) \to [0, \infty)\) is such that the following exist:

\[f_0 := \lim_{y \to 0^+} \frac{f(\cdot, y)}{y}, \quad f_\infty := \lim_{y \to \infty} \frac{f(\cdot, y)}{y};\]
and the right-dense continuous function $h : [t_1, t_n] \rightarrow [0, \infty)$ satisfies some suitable conditions to be developed. Problem (1.1), (1.2) is an extension of the three-point problems discussed in Sun and Li [1, 2], Anderson [3], Atici and Guseinov [4], Kaufmann [5], Kaufmann and Raffoul [6]. Other related three-point problems on time scales include Anderson, Avery, and Henderson [7], Peterson, Raffoul, and Tisdell [8], and a singular case in DaCunha, Davis, and Singh [9]. Some of the work on multi-point time-scale problems includes Anderson [10, 11] and Kong and Kong [12]. See also Ma [13] and Ma and Thompson [14] for related results when $T = \mathbb{R}$. Other work on the existence of solutions to dynamic equations on time scales (measure chains) includes Chyan and Henderson [15], Erbe and Peterson [16, 17], and Henderson [18]. For more general information concerning dynamic equations on time scales, introduced by Aulbach and Hilger [19] and Hilger [20], see the excellent text by Bohner and Peterson [21] and their edited text [22].

2. FOUNDATIONAL LEMMAS

We first construct the Green's function for the second-order boundary value problem

\begin{align*}
(\Delta (py^{\nabla^2})) (t) - q(t)y(t) + u(t) &= 0, \quad t_1 < t < t_n, \\
\alpha y(t_1) - \beta p(t_1)y^{\nabla}(t_1) &= 0, \quad \gamma y(t_n) + \delta p(t_n)y^{\nabla}(t_n) = 0,
\end{align*}

where $\alpha, \beta, \gamma, \delta$ are real numbers such that $|\alpha| + |\beta| \neq 0, |\gamma| + |\delta| \neq 0$. The techniques here are similar to those found in [4].

Denote by $\phi$ and $\psi$ the solutions of the corresponding homogeneous equation

\begin{equation}
(\Delta (py^{\nabla^2})) (t) - q(t)y(t) = 0, \quad t \in [t_1, t_n),
\end{equation}

under the initial conditions

\begin{align*}
\psi(t_1) &= \beta, \quad p(t_1)\psi^{\nabla}(t_1) = \alpha, \\
\phi(t_n) &= \delta, \quad p(t_n)\phi^{\nabla}(t_n) = -\gamma,
\end{align*}

so that $\psi$ and $\phi$ satisfy the first and second boundary conditions in (2.2), respectively. Set

\begin{equation}
d = -W_t(\psi, \phi) = p(t)\psi^{\nabla}(t)\phi(t) - \psi(t)p(t)\phi^{\nabla}(t).
\end{equation}

Since the Wronskian of any two solutions is independent of $t$, evaluating at $t = t_1$, $t = t_n$, and using the boundary conditions (2.4), (2.5) yields

\begin{equation}
d = \alpha \phi(t_1) - \beta p(t_1)\phi^{\nabla}(t_1) = \gamma \psi(t_n) + \delta p(t_n)\psi^{\nabla}(t_n).
\end{equation}

In addition $d \neq 0$ if and only if the homogeneous equation (2.3) has only the trivial solution satisfying the boundary conditions (2.2). For the proof of the following lemma, see Theorem 4.2 and Remark 4.1 in [4].
Lemma 2.1. Assume (1.3) and (1.4) hold. If \(d \neq 0\), then the nonhomogeneous boundary value problem (2.1), (2.2), has a unique solution \(y\) for which the formula

\[
y(t) = \int_{t_1}^{t_n} G(t, s)u(s)\Delta s, \quad t \in [\rho(t_1), t_n]
\]

holds, where the function \(G(t, s)\) is given by

\[
G(t, s) = \frac{1}{d} \begin{cases} 
\psi(t)\phi(s), & \rho(t_1) \leq t \leq s \leq t_n \\
\phi(s)\psi(t), & \rho(t_1) \leq s \leq t \leq t_n,
\end{cases}
\]

and \(G(t, s)\) is the Green’s function of the boundary value problem (2.1), (2.2).

Lemma 2.2. Assume (1.3) and (1.4) hold. Then the functions \(\psi\) and \(\phi\) satisfy

\[
\psi(t) \geq 0, \quad t \in [\rho(t_1), t_n], \quad \psi(t) > 0, \quad t \in (\rho(t_1), t_n], \quad p(t)\psi^\Delta(t) \geq 0, \quad t \in [\rho(t_1), t_n],
\]

\[
\phi(t) \geq 0, \quad t \in [\rho(t_1), t_n], \quad \phi(t) > 0, \quad t \in [\rho(t_1), t_n), \quad p(t)\phi^\Delta(t) \leq 0, \quad t \in [\rho(t_1), t_n].
\]

Proof. The proof is very similar to the proof of Lemma 5.1 in [4] and is omitted. \(\square\)

Set

\[
D := \begin{vmatrix} 
-\sum_{i=2}^{n-1} a_i\psi(t_i) & d - \sum_{i=2}^{n-1} a_i\phi(t_i) \\
d - \sum_{i=2}^{n-1} b_i\psi(t_i) & -\sum_{i=2}^{n-1} b_i\phi(t_i)
\end{vmatrix}.
\]

Lemma 2.3. Assume (1.3) and (1.4) hold. If \(D \neq 0\) and \(u \in C_{rd}[t_1, t_n]\), then the nonhomogeneous dynamic equation (2.1) with boundary conditions (1.2) has a unique solution \(y\) for which the formula

\[
y(t) = \int_{t_1}^{t_n} G(t, s)u(s)\Delta s + A(u)\psi(t) + B(u)\phi(t), \quad t \in [\rho(t_1), t_n]
\]

holds, where the function \(G(t, s)\) is the Green’s function (2.7) of the boundary value problem (2.1), (2.2), and the functionals \(A\) and \(B\) are defined by

\[
A(u) := \frac{1}{D} \begin{vmatrix} 
\sum_{i=2}^{n-1} a_i \int_{t_1}^{t_n} G(t_i, s)u(s)\Delta s & d - \sum_{i=2}^{n-1} a_i\phi(t_i) \\
d - \sum_{i=2}^{n-1} b_i\psi(t_i) & -\sum_{i=2}^{n-1} b_i\phi(t_i)
\end{vmatrix},
\]

\[
B(u) := \frac{1}{D} \begin{vmatrix} 
-\sum_{i=2}^{n-1} a_i\psi(t_i) & \sum_{i=2}^{n-1} a_i \int_{t_1}^{t_n} G(t_i, s)u(s)\Delta s \\
-\sum_{i=2}^{n-1} b_i\psi(t_i) & \sum_{i=2}^{n-1} b_i \int_{t_1}^{t_n} G(t_i, s)u(s)\Delta s
\end{vmatrix}.
\]

Proof. It can be verified that for a solution \(y\) of the nonhomogeneous equation (2.1) under the nonhomogeneous boundary conditions (1.2), the formula (2.9) holds, where \(G(t, s)\) is given by (2.7). We thus show that the function \(y\) given in (2.9) is a solution of (2.1) with conditions (1.2) only if \(A\) and \(B\) are given by (2.10) and (2.11), respectively. If \(y\) as in (2.9) is a solution of (2.1), (1.2), then

\[
y(t) = \frac{1}{d} \int_{t_1}^{t} \psi(t)\phi(s)u(s)\Delta s + \frac{1}{d} \int_{t}^{t_n} \psi(t)\phi(s)u(s)\Delta s + A\psi(t) + B\phi(t)
\]
for some constants $A$ and $B$. Taking the nabla derivative and multiplying by $p$ yields
\[ py^\nabla = \frac{p\phi^\nabla}{d} \int_{t_1}^{t} \psi(s)u(s)\Delta s + \frac{p\psi^\nabla}{d} \int_{t}^{t_n} \phi(s)u(s)\Delta s + Ap\psi^\nabla + Bp\phi^\nabla; \]

the delta derivative of this expression is
\[ (py^\nabla)^\Delta = \left( \frac{p\phi^\nabla}{d} \right)^\Delta \int_{t_1}^{\sigma(t)} \psi(s)u(s)\Delta s + \phi(t)u(t) + A(p\psi^\nabla)^\Delta + B(p\phi^\nabla)^\Delta \]
\[ + \left( \frac{p\psi^\nabla}{d} \right)^\Delta \int_{\sigma(t)}^{t_n} \phi(s)u(s)\Delta s - \frac{p\psi^\nabla}{d} \phi(t)u(t). \]

Using Theorem 1.75 from [21], and the fact that $\psi$ and $\phi$ are solutions to (2.3), we obtain
\[ (py^\nabla)^\Delta(t) = \frac{q(t)}{d} \int_{t_1}^{t} \phi(t)\psi(s)u(s)\Delta s + \frac{q(t)}{d} \phi(t)\mu_\sigma(t)\psi(t)u(t) + \frac{u(t)}{d} p(t)\phi^\nabla(t)\phi(t) \]
\[ + \frac{q(t)}{d} \int_{t}^{t_n} \psi(s)\phi(s)u(s)\Delta s - \frac{q(t)}{d} \phi(s)\mu_\sigma(t)\phi(t)u(t) \]
\[ - \frac{u(t)}{d} p(t)\psi^\nabla(t)\phi(t) + q(t)(A\psi(t) + b\phi(t)). \]

Recall that $d$ is in terms of the Wronskian of $\psi$ and $\phi$ in (2.6); it follows that
\[ (py^\nabla)^\Delta(t) = q(t)y(t) - u(t). \]

Now
\[ y(t_1) = \frac{\psi(t_1)}{d} \int_{t_1}^{t_n} \phi(s)u(s)\Delta s + A\psi(t_1) + B\phi(t_1) \]
and
\[ p(t_1)y^\nabla(t_1) = \frac{p(t_1)\psi^\nabla(t_1)}{d} \int_{t_1}^{t_n} \phi(s)u(s)\Delta s + Ap(t_1)\psi^\nabla(t_1) + Bp(t_1)\phi^\nabla(t_1); \]

multiply the first line by $\alpha$ and the second by $-\beta$, and use (1.2) and (2.4) to see that
\[ B \left[ \alpha\phi(t_1) - \beta p(t_1)\phi^\nabla(t_1) \right] = \sum_{i=2}^{n-1} a_i \left( \int_{t_1}^{t_n} G(t_i, s)u(s)\Delta s + A\psi(t_i) + B\phi(t_i) \right). \]

At the other end,
\[ y(t_n) = \frac{\phi(t_n)}{d} \int_{t_1}^{t_n} \psi(s)u(s)\Delta s + A\psi(t_n) + B\phi(t_n) \]
and
\[ p(t_n)y^\nabla(t_n) = \frac{p(t_n)\phi^\nabla(t_n)}{d} \int_{t_1}^{t_n} \psi(s)u(s)\Delta s + Ap(t_n)\psi^\nabla(t_n) + Bp(t_n)\phi^\nabla(t_n); \]

consequently
\[ (2.13) \]
\[ A \left[ \gamma\psi(t_n) + \delta p(t_n)\psi^\nabla(t_n) \right] = \sum_{i=2}^{n-1} b_i \left( \int_{t_1}^{t_n} G(t_i, s)u(s)\Delta s + A\psi(t_i) + B\phi(t_i) \right). \]
Combining (2.12) and (2.13) and using (2.6), we arrive at the system of equations

\[- A \sum_{i=2}^{n-1} a_i \psi(t_i) + B \left[ \alpha \phi(t_1) - \beta p(t_1) \phi(t_1) - \sum_{i=2}^{n-1} a_i \phi(t_i) \right] = \sum_{i=2}^{n-1} a_i \int_{t_1}^{t_i} G(t_i, s) u(s) \Delta s \]

\[ A \left[ \gamma \psi(t_n) + \delta p(t_n) \psi(t_n) - \sum_{i=2}^{n-1} b_i \psi(t_i) \right] - B \sum_{i=2}^{n-1} b_i \phi(t_i) = \sum_{i=2}^{n-1} b_i \int_{t_1}^{t_n} G(t_i, s) u(s) \Delta s. \]

Again using (2.6) at both \( t_1 \) and \( t_n \), we verify (2.10) and (2.11) hold. \( \square \)

**Lemma 2.4.** Let (1.3) and (1.4) hold, and assume

\[(2.14) \quad D < 0, \quad d - \sum_{i=2}^{n-1} a_i \phi(t_i) > 0, \quad d - \sum_{i=2}^{n-1} b_i \psi(t_i) > 0 \]

for \( D \) and \( d \) given in (2.8) and (2.6), respectively. If \( u \in C_{rd}[t_1, t_n] \) with \( u \geq 0 \), the unique solution \( y \) as in (2.9) of the problem (2.1), (1.2) satisfies \( y(t) \geq 0 \) for \( t \in [t_1, t_n] \).

**Proof.** From the previous lemmas and assumptions, we know that the Green’s function (2.7) satisfies \( G(t, s) \geq 0 \) on \([\rho(t_1), t_n] \times [\rho(t_1), t_n]\). Hypotheses (1.3), (1.4), and (2.14) applied to (2.10) and (2.11) imply that \( A(u), B(u) \geq 0 \). \( \square \)

Suppose (2.14) does not hold. For example, let \( n = 3, p(t) \equiv 1 = \alpha = \gamma, q(t) \equiv 0 = \beta = \delta = a_2, \) and \( t_1 = 0 \). Then (2.1), (1.2) becomes

\[ y^{\nabla \Delta}(t) + u(t) = 0, \quad t_1 < t < t_3, \quad y(t_1) = 0, \quad y(t_3) = b_2 y(t_2). \]

Note that \( \psi(t) = t, d = t_3, \) and \( D = t_3(b_2 t_2 - t_3) \). If \( D > 0 \), then \( b_2 t_2 > t_3 \), and there is no positive solution; see Lemma 5 in [3].

**Lemma 2.5.** Let (1.3), (1.4), and (2.14) hold, and fix

\[ \xi_1, \xi_2 \in \mathbb{T}^\kappa, \quad \rho(t_1) < \xi_1 < \xi_2 < t_n. \]

If \( u \in C_{rd}[t_1, t_n] \) with \( u \geq 0 \), the unique solution \( y \) as in (2.9) of the time scale boundary value problem (2.1), (1.2) satisfies

\[ \min_{t \in [\xi_1, \xi_2]} y(t) \geq \Gamma \| y \|, \quad \| y \| := \max_{t \in [\rho(t_1), t_n]} y(t), \]

where

\[(2.15) \quad \Gamma := \min \left\{ \frac{\phi(\xi_2)}{\phi(\rho(t_1))}, \frac{\psi(\xi_1)}{\psi(t_n)} \right\} \in (0, 1). \]
Proof. From (1.3), (2.7), and Lemma 2.2,

\begin{equation}
0 \leq G(t, s) \leq G(s, s), \quad t \in [\rho(t_1), t_n],
\end{equation}

so that

\begin{equation}
y(t) \leq \int_{t_1}^{t_n} G(s, s)u(s)\Delta s + A(u)\psi(t_n) + B(u)\phi(\rho(t_1)), \quad \forall t \in [\rho(t_1), t_n].
\end{equation}

For \( t \in [\xi_1, \xi_2], \) the Green’s function (2.7) satisfies

\begin{equation}
\frac{G(t, s)}{G(s, s)} = \begin{cases} 
\frac{\phi(t)}{\phi(s)} : \rho(t_1) \leq s \leq t \leq t_n \\
\frac{\psi(t)}{\psi(s)} : \rho(t_1) \leq t \leq s \leq t_n 
\end{cases} \geq \Gamma, 
\end{equation}

for \( \Gamma \) as in (2.15), and

\begin{equation}
y(t) = \int_{t_1}^{t_n} \frac{G(t, s)}{G(s, s)} G(s, s)u(s)\Delta s + A(u)\psi(t) + B(u)\phi(t) \\
\geq \int_{t_1}^{t_n} \Gamma G(s, s)u(s)\Delta s + A(u)\psi(\xi_1) + B(u)\phi(\xi_2) \\
\geq \Gamma \left( \int_{t_1}^{t_n} G(s, s)u(s)\Delta s + A(u)\psi(t_n) + B(u)\phi(\rho(t_1)) \right) \geq \Gamma\|y\|.
\end{equation}

\[ \Box \]

3. RESULTS ON CONES

Assume the right-dense continuous function \( h \) satisfies

\begin{equation}
h : [t_1, t_n] \to [0, \infty) \quad \text{and there exists} \quad t_\ast \in (\sigma(t_1), \rho(t_n)) \quad \text{such that} \quad h(t_\ast) > 0.
\end{equation}

Then there exist \( \xi_1, \xi_2 \) as in Lemma 2.5 such that

\[ \xi_1 < t_\ast < \xi_2, \quad \int_{\xi_1}^{\xi_2} G(t, s)h(s)\Delta s > 0, \quad t \in (\rho(t_1), t_n). \]

In the following, let \( \Gamma \) be the constant defined in (2.15) with respect to such constants \( \xi_1, \xi_2. \) Let \( \tau \in [\xi_1, \xi_2] \) be determined by

\begin{equation}
\int_{\xi_1}^{\xi_2} G(\tau, s)h(s)\Delta s = \min_{t \in [\xi_1, \xi_2]} \int_{\xi_1}^{\xi_2} G(t, s)h(s)\Delta s > 0.
\end{equation}

For \( G(t, s) \) in (2.7) and \( A, B \) as in (2.10), (2.11), respectively, define the constant

\begin{equation}
K := \int_{t_1}^{t_n} G(s, s)h(s)\Delta s + A(h)\psi(t_n) + B(h)\phi(\rho(t_1)).
\end{equation}

Let \( S \) denote the Banach space \( C[\rho(t_1), t_n] \) with the norm \( \|y\| = \sup_{t \in [\rho(t_1), t_n]} |y(t)|. \)

Define the cone \( P \subset S \) by

\begin{equation}
P = \{ y \in S : y(t) \geq 0 \text{ on } [\rho(t_1), t_n] \text{ and } y(t) \geq \Gamma\|y\| \text{ on } [\xi_1, \xi_2] \},
\end{equation}
where $\Gamma$ is given in (2.15). Since $y$ is a solution of (1.1), (1.2) if and only if
\[
y(t) = \int_{t_1}^{t_n} G(t, s) h(s) f(s, y(s)) \Delta s + A(hf(\cdot, y)) \psi(t) + B(hf(\cdot, y)) \phi(t), \quad t \in [\rho(t_1), t_n],
\]
define, for $y \in \mathcal{P}$, the operator $T : \mathcal{P} \to \mathcal{S}$ by
\[
(Ty)(t) := \int_{t_1}^{t_n} G(t, s) h(s) f(s, y(s)) \Delta s + A(hf(\cdot, y)) \psi(t) + B(hf(\cdot, y)) \phi(t).
\]
If $y \in \mathcal{P}$, then by (2.16) we have
\[
(Ty)(t) = \int_{t_1}^{t_n} G(t, s) h(s) f(s, y(s)) \Delta s + A(hf(\cdot, y)) \psi(t) + B(hf(\cdot, y)) \phi(t)
\leq \int_{t_1}^{t_n} G(s, s) h(s) f(s, y(s)) \Delta s + A(hf(\cdot, y)) \psi(t_n) + B(hf(\cdot, y)) \phi(\rho(t_1)),
\]
so that for $t \in [\xi_1, \xi_2]$,
\[
(Ty)(t) = \int_{t_1}^{t_n} G(t, s) h(s) f(s, y(s)) \Delta s + A(hf(\cdot, y)) \psi(t) + B(hf(\cdot, y)) \phi(t)
\geq \int_{t_1}^{t_n} \frac{G(t, s)}{G(s, s)} G(s, s) h(s) f(s, y(s)) \Delta s + A(hf(\cdot, y)) \psi(\xi_1) + B(hf(\cdot, y)) \phi(\xi_2)
\geq \Gamma \left( \int_{t_1}^{t_n} G(s, s) h(s) f(s, y(s)) \Delta s + A(hf(\cdot, y)) \psi(t_n) + B(hf(\cdot, y)) \phi(\rho(t_1)) \right)
\geq \Gamma \|Ty\|.
\]
Therefore $T : \mathcal{P} \to \mathcal{P}$. Moreover, $T$ is completely continuous by a typical application of the Ascoli-Arzela Theorem.

**Lemma 3.1.** [23, 24] Let $P$ be a cone in a Banach space $S$, and let $B$ be an open, bounded subset of $S$ with $B_P := B \cap P \neq \emptyset$ and $\overline{B_P} \neq P$. Assume that $T : \overline{B_P} \to P$ is a compact map such that $y \neq Ty$ for $y \in \partial B_P$. The following results hold.

(i) If $\|Ty\| \leq \|y\|$ for $y \in \partial B_P$, then $i_P(T, B_P) = 1$.

(ii) If there exists an $\eta \in P \setminus \{0\}$ such that $y \neq Ty + \lambda \eta$ for all $y \in \partial B_P$ and all $\lambda > 0$, then $i_P(T, B_P) = 0$.

(iii) Let $U$ be open in $P$ such that $\overline{U} \subset B_P$. If $i_P(T, B_P) = 1$ and $i_P(T, U_P) = 0$, then $T$ has a fixed point in $B_P \setminus \overline{U_P}$; the same is true if $i_P(T, B_P) = 0$ and $i_P(T, U_P) = 1$.

For the cone $\mathcal{P}$ given above in (3.4) and any positive real number $r$, define the convex set
\[
P_r := \{ y \in \mathcal{P} : \|y\| < r \},
\]
and, for $\Gamma$ in (2.15), the set
\[
\Omega_r := \left\{ y \in \mathcal{P} : \min_{t \in [\xi_1, \xi_2]} y(t) < \Gamma r \right\}.
\]
Lemma 3.2. [23] The set $\Omega_r$ has the following properties.

(i) $\Omega_r$ is open relative to $\mathcal{P}$.

(ii) $P_{\Gamma r} \subset \Omega_r \subset P_r$.

(iii) $y \in \partial \Omega_r$ if and only if $\min_{t \in [\xi_1, \xi_2]} y(t) = \Gamma r$.

(iv) If $y \in \partial \Omega_r$, then $\Gamma r \leq y(t) \leq r$ for $t \in [\xi_1, \xi_2]$.

As in [2], we introduce the following notation for further reference. Let

$$f^r_{\Gamma r} := \min \left\{ \min_{t \in [\xi_1, \xi_2]} \frac{f(t, y)}{r} : y \in [\Gamma r, r] \right\},$$

$$f^r_0 := \max \left\{ \max_{t \in [\rho(t_1), t_n]} \frac{f(t, y)}{r} : y \in [0, r] \right\},$$

$$f^a := \limsup_{y \to a} \max_{t \in [\rho(t_1), t_n]} \frac{f(t, y)}{y},$$

$$f_a := \liminf_{y \to a} \min_{t \in [\xi_1, \xi_2]} \frac{f(t, y)}{y} \quad (a := 0^+, \infty).$$

In the next two lemmas, we sanction conditions on $f$ guaranteeing that $i_P(T, P_r) = 1$ or $i_P(T, \Omega_r) = 0$.

Lemma 3.3. Let $K$ be as in (3.3). If the conditions

$$f^r_0 \leq 1/K \quad \text{and} \quad y \neq Ty \quad \text{for} \quad y \in \partial P_r$$

hold, then $i_P(T, P_r) = 1$.

Proof. From (2.10) and (2.11),

$$|A(hf(\cdot, y))| \leq A(h)\|f(\cdot, y)\|, \quad |B(hf(\cdot, y))| \leq B(h)\|f(\cdot, y)\|.$$ 

Let $y \in \partial P_r$; using (2.16) we have

$$(Ty)(t) = \int_{t_1}^{t_n} G(t, s)h(s)f(s, y(s))\Delta s + A(hf(\cdot, y))\psi(t) + B(hf(\cdot, y))\phi(t)$$

$$\leq \|f(\cdot, y)\| \left( \int_{t_1}^{t_n} G(s, s)h(s)\Delta s + A(h)\psi(t_n) + B(h)\phi(\rho(t_1)) \right)$$

$$\leq (r/K)K = r = \|y\|.$$ 

It follows that for $y \in \partial P_r$, $\|Ty\| \leq \|y\|$. By Lemma 3.1 (i), $i_P(T, P_r) = 1$. \hfill \square

Lemma 3.4. Let $\tau$ be as in (3.2), and let

$$M^{-1} := \int_{\xi_1}^{\xi_2} G(\tau, s)h(s)\Delta s.$$ 

If the conditions

$$f^r_{\Gamma r} \geq M\Gamma \quad \text{and} \quad y \neq Ty \quad \text{for} \quad y \in \partial \Omega_r$$

hold, then $i_P(T, \Omega_r) = 0$. 

Proof. Let \( \eta(t) \equiv 1 \) for \( t \in [\rho(t_1), t_n] \), so that \( \eta \in \partial P_1 \). Suppose there exist \( y_* \in \partial \Omega_r \) and \( \lambda_* > 0 \) such that \( y_* = Ty_* + \lambda_* \eta \). Then for \( t \in [\xi_1, \xi_2] \),

\[
y_* (t) = (Ty_*)(t) + \lambda_* \eta = \int_{\xi_1}^{\xi_2} G(t, s)\eta(s) f(s, y(s)) ds + \lambda_* \geq M \Gamma r \int_{\xi_1}^{\xi_2} G(t, s)\eta(s) ds + \lambda_* = \Gamma r + \lambda_*.
\]

But this implies that \( \Gamma r \geq \Gamma r + \lambda_* \), a contradiction. Consequently, \( y_* \neq Ty_* + \lambda_* \eta \) for \( y_* \in \partial \Omega_r \) and \( \lambda_* > 0 \), so by Lemma 3.1 (ii) we have \( i_\Gamma (T, \Omega_r) = 0 \). \( \square \)

4. ONE OR TWO SOLUTIONS

In what follows, we seek fixed points in the cone \( P \) of \( T \) given in (3.5) using the lemmas established in the previous sections; these fixed points will be positive solutions to the multi-point boundary value problem (1.1), (1.2).

Theorem 4.1. Let \( \Gamma, K, \) and \( M \) be as given in (2.15), (3.3), and (3.6), respectively. Assume that there exist constants \( c_1, c_2, c_3 \in \mathbb{R} \) with \( 0 < c_1 < \Gamma c_2 \) and \( c_2 < c_3 \) such that

\[
\begin{align*}
(\text{H1}) & \quad f_0^{c_1}, f_0^{c_3} \leq 1/K, & f_{\Gamma c_2}^{c_2} \geq M \Gamma, \quad \text{and} \quad y \neq Ty \quad \text{for} \quad y \in \partial \Omega_{c_2}, \\
(\text{H2}) & \quad f_{\Gamma c_1}^{c_1}, f_{\Gamma c_3}^{c_3} \geq M \Gamma, & f_0^{c_2} \leq 1/K, \quad \text{and} \quad y \neq Ty \quad \text{for} \quad y \in \partial P_{c_2}.
\end{align*}
\]

Then (1.1), (1.2) has two positive solutions in \( P \) as in (3.4). Additionally, if in (H1) the condition \( f_0^{c_1} \leq 1/K \) is replaced by \( f_0^{c_1} \leq 1/K \), then (1.1), (1.2) has a third positive solution in \( P_{c_1} \).

Proof. Assume (H2) holds; the case for (H1) is similar and is omitted. We show that either \( T \) has a fixed point in \( \partial \Omega_{c_1} \) or in \( P_{c_2} \setminus \Omega_{c_1} \). From Lemma 3.4, if \( y \neq Ty \) for \( y \in \partial \Omega_{c_1} \cup \partial \Omega_{c_3} \), then \( i_\Gamma (T, \Omega_{c_1}) = 0 \) and \( i_\Gamma (T, \Omega_{c_3}) = 0 \). Since \( f_0^{c_2} \leq 1/K \) and \( y \neq Ty \) for \( y \in \partial P_{c_2} \), Lemma 3.3 implies that \( i_\Gamma (T, P_{c_2}) = 1 \). By Lemma 3.2 (ii), \( \Omega_{c_1} \subset P_{c_1} \subset P_{c_2} \). From Lemma 3.1 (iii), \( T \) has a fixed point in \( P_{c_2} \setminus \Omega_{c_1} \). In the same way \( P_{c_2} \subset P_{\Gamma c_3} \subset \Omega_{c_3} \) and \( T \) has a fixed point in \( \Omega_{c_3} \setminus \Omega_{c_2} \). \( \square \)

Corollary 4.2. Suppose there exists a constant \( c > 0 \) such that either

\[
\begin{align*}
(\text{H1'}) & \quad 0 \leq f_0, f_\infty < 1/K, & f_{\Gamma c} \geq M \Gamma, \quad \text{and} \quad y \neq Ty \quad \text{for} \quad y \in \partial \Omega_c \\
(\text{H2'}) & \quad M < f_0, f_\infty \leq \infty, & f_0 \leq 1/K, \quad \text{and} \quad y \neq Ty \quad \text{for} \quad y \in \partial P_c
\end{align*}
\]

holds. Then (1.1), (1.2) has two positive solutions in \( P \).

Proof. Since (H1') implies (H1) and (H2') implies (H2), the result follows. \( \square \)
The proofs of the following two results are similar to those given above and are omitted.

**Theorem 4.3.** Assume that there exist constants \( c_1, c_2 \in \mathbb{R} \) with \( 0 < c_1 < \Gamma c_2 \) such that
\[
(H3) \quad f_0^{c_1} \leq 1/K \quad \text{and} \quad f_{\Gamma c_2}^{c_2} \geq M\Gamma,
\]
or that there exist constants \( c_1, c_2 \in \mathbb{R} \) with \( 0 < c_1 < c_2 \) such that
\[
(H4) \quad f_{\Gamma c_1}^{c_1} \geq M\Gamma \quad \text{and} \quad f_0^{c_2} \leq 1/K.
\]
Then (1.1), (1.2) has a positive solution.

**Corollary 4.4.** Suppose either
\[
(H3') \quad 0 \leq f^0 < 1/K \quad \text{and} \quad M < f_\infty \leq \infty
\]
or
\[
(H4') \quad 0 \leq f_\infty < 1/K \quad \text{and} \quad M < f_0 \leq \infty
\]
holds. Then (1.1), (1.2) has a positive solution.

## 5. THREE SOLUTIONS

To prove the existence of at least three positive solutions to (1.1), (1.2), we will use the Leggett-Williams fixed point theorem [24, 25]:

**Theorem 5.1.** Let \( P \) be a cone in the real Banach space \( \mathcal{S} \), \( T : \overline{P}_c \rightarrow \overline{P}_c \) be completely continuous and \( \varpi \) be a nonnegative continuous concave functional on \( P \) with \( \varpi(y) \leq \|y\| \) for all \( y \in \overline{P}_c \). Suppose there exist constants \( 0 < a < b < \ell \leq c \) such that the following conditions hold:

(i) \( \{y \in P(\varpi, b, \ell) : \varpi(y) > b\} \neq \emptyset \) and \( \varpi(Ty) > b \) for all \( y \in P(\varpi, b, \ell) \);
(ii) \( \|Ty\| < a \) for \( \|y\| \leq a \);
(iii) \( \varpi(Ty) > b \) for \( y \in P(\varpi, b, c) \) with \( \|Ty\| > \ell \).

Then \( T \) has at least three fixed points \( y_1, y_2, \) and \( y_3 \) in \( \overline{P}_c \) satisfying:
\[
\|y_1\| < a, \quad \varpi(y_2) > b, \quad a < \|y_3\| \quad \text{with} \quad \varpi(y_3) < b.
\]

Define the continuous concave functional \( \varpi : \mathcal{P} \rightarrow [0, \infty) \) by \( \varpi(y) := \min_{t \in [\xi_1, \xi_2]} y(t) \).

Moreover, we take
\[
P_c = \{y \in \mathcal{P} : \|y\| < c\}, \quad P(\varpi, a, b) := \{y \in \mathcal{P} : a \leq \varpi(y), \|y\| \leq b\}.
\]

**Theorem 5.2.** Take \( \xi_1, \xi_2 \) as in Lemma 2.5, \( \Gamma \) as in (2.15), \( K \) as in (3.3), and \( M \) as in (3.6). Assume there exist constants \( 0 < a < b \) such that the following conditions hold:
(i) \( f(t, y) \geq bM \) for all \( t \in [\xi_1, \xi_2] \), \( y \in [b, b/\Gamma] \);
(ii) \( f(t, y) < a/K \) for all \( t \in [\rho(t_1), t_n] \), \( y \in [0, a] \);
(iii) one of the following is satisfied:
\begin{enumerate}
  \item \( \lim_{y \to \infty} \max_{t \in [\rho(t_1), t_n]} \frac{f(t, y)}{y} < \frac{1}{K} \);
  \item there exists a constant \( c > a/\Gamma \) such that \( f(t, y) \leq c/K \) for \( t \in [\rho(t_1), t_n] \) and \( y \in [0, c] \).
\end{enumerate}

Then (1.1), (1.2) has at least three positive solutions \( y_1, y_2, \) and \( y_3 \) in \( \mathcal{T} \) satisfying:
\[
\|y_1\| < a, \quad \min_{t \in [\xi_1, \xi_2]} y_2(t) > b, \quad a < \|y_3\| \quad \text{with} \quad \min_{t \in [\xi_1, \xi_2]} y_3(t) < b.
\]

**Proof.** We seek fixed points in the cone \( \mathcal{P} \), given in (3.4), of \( T \) as in (3.5) using Theorem 5.1 above. Throughout the proof we set \( \ell := b/\Gamma \). The technique here is similar to that used in a third-order discrete case in [26].

Claim 1: If condition (A) holds, then there exists a number \( c \) such that \( c > \ell \) and \( T : \mathcal{P} \to \mathcal{P} \).

If
\[
\limsup_{y \to \infty} \max_{t \in [\rho(t_1), t_n]} \frac{f(t, y)}{y} < \frac{1}{K},
\]
then there exists a \( \theta > 0 \) and \( \epsilon < \frac{1}{K} \) such that if \( y > \theta \), then \( \max_{t \in [\rho(t_1), t_n]} \frac{f(t, y)}{y} < \epsilon \).

For \( \lambda := \max\{f(t, y) : y \in [0, \theta], t \in [\rho(t_1), t_n]\} \), we have \( f(t, y) \leq \epsilon y + \lambda \) for all \( y \geq 0 \), for all \( t \in [\rho(t_1), t_n] \). Pick any
\[
c > \max \left\{ \ell, \frac{\lambda}{1/K - \epsilon} \right\}.
\]

Then \( y \in \mathcal{P} \) implies that
\[
\|Ty\| = \max_{t \in [\rho(t_1), t_n]} \int_{t_1}^{t_n} G(t, s) h(s) f(s, y(s)) \Delta s + A(hf(\cdot, y)) \psi(t) + B(hf(\cdot, y)) \phi(t)
\]
\[
\leq (\epsilon \|y\| + \lambda) \left( \max_{t \in [\rho(t_1), t_n]} \int_{t_1}^{t_n} G(t, s) h(s) \Delta s + A(h \psi(t) + B(h \phi(t) \right)
\]
\[
\leq (\epsilon \|y\| + \lambda) K < c \epsilon K + c(1 - c K) = c.
\]

Thus, \( T : \mathcal{P} \to \mathcal{P} \) and Claim 1 has been shown.

Claim 2: If there exists a positive number \( r \) such that \( y \in [0, r] \) implies \( f(t, y) < r/K \) for \( t \in [\rho(t_1), t_n] \), then \( T : \mathcal{P} \to \mathcal{P} \).

If \( y \in \mathcal{P} \), then
\[
\|Ty\| = \max_{t \in [\rho(t_1), t_n]} \int_{t_1}^{t_n} G(t, s) h(s) f(s, y(s)) \Delta s + A(hf(\cdot, y)) \psi(t) + B(hf(\cdot, y)) \phi(t)
\]
\[
\leq (r/K) \left( \max_{t \in [\rho(t_1), t_n]} \int_{t_1}^{t_n} G(t, s) h(s) \Delta s + A(h \psi(t) + B(h \phi(t) \right) \leq r.
\]
As a result, \( T: \mathcal{P}_r \to P_r \) and Claim 2 has been shown. We have thus seen in the two previous claims that if either condition (A) or condition (B) holds, then there exists a number \( c \) such that \( c > \ell \) and \( T: \mathcal{P}_c \to P_c \). Note from Claim 2 with \( r = a \) we get, using hypothesis (ii), that \( T: \mathcal{P}_a \to P_a \).

Claim 3: \( \{ y \in P(\varpi, b, \ell) : \varpi(y) > b \} \neq \emptyset \) and \( \varpi(Ty) > b \) for all \( y \in P(\varpi, b, \ell) \).

It is easy to see that \( y(t) \equiv (\ell + b)/2 \in \{ y \in P(\varpi, b, \ell) : \varpi(y) > b \} \neq \emptyset \). Next, let \( y \in P(\varpi, b, \ell), \tau \) as in (3.2), and \( M \) as in (3.6). Using (i),

\[
\varpi(Ty) = \min_{t \in [\xi_1, \xi_2]} \int_{t_1}^{t_n} G(t, s)h(s)f(s, y(s))\Delta s + A(hf(\cdot, y))\psi(t) + B(hf(\cdot, y))\phi(t) > \min_{t \in [\xi_1, \xi_2]} \int_{t_1}^{\xi_2} G(t, s)h(s)f(s, y(s))\Delta s \geq bM/M = b,
\]

by the definition of \( \ell \). Consequently, \( y \in P(\varpi, b, \ell) \) yields \( \varpi(Ty) > b \).

Claim 4: If \( y \in P(\varpi, b, c) \) and \( ||Ty|| > \ell \), then \( \varpi(Ty) > b \).

Let \( \Gamma \) be as in (2.15) and recall that \( T: \mathcal{P} \to \mathcal{P} \). If \( y \in P(\varpi, b, c) \) and \( ||Ty|| > \ell \), then

\[
\varpi(Ty) = \min_{t \in [\xi_1, \xi_2]} (Ty)(t) \geq \Gamma ||Ty|| > \Gamma \ell = b.
\]

Since all of the hypotheses of the Leggett-Williams existence theorem are satisfied, (1.1), (1.2) has at least three positive solutions. \( \square \)

**Corollary 5.3.** Suppose that there exist constants

\[
0 < \alpha_1 < \beta_1 < \ell_1 < \alpha_2 < \beta_2 < \ell_2 < \alpha_3 < \cdots < \alpha_m, \quad m \in \mathbb{N},
\]

where \( \ell_i := \beta_i/\Gamma \), such that the following conditions hold:

(i') \( f(t, y) \geq \beta_i M \) for all \( t \in [\xi_1, \xi_2], \ y \in [\beta_i, \ell_i] \);

(ii') \( f(t, y) < \alpha_i/K \) for all \( t \in [\rho(t_1), t_n], \ y \in [0, \alpha_i] \).

Then (1.1), (1.2) has at least \( 2m - 1 \) positive solutions.

**REFERENCES**


