MULTIPLE PERIODIC SOLUTIONS FOR A SECOND-ORDER PROBLEM ON PERIODIC TIME SCALES

DOUGLAS R. ANDERSON

Abstract. Green’s function for a second-order delta dynamic equation is found, then used as the kernel in an integral operator to guarantee the existence of positive periodic solutions to a second-order periodic boundary value problem with periodic coefficient functions.

1. Introduction to the Periodic Problem

Throughout this work we assume a working knowledge of time scales and time-scale notation; please see Hilger [1], Agarwal and Bohner [2], or the recent books by Bohner and Peterson [3, 4] and Kaymakçalan, Lakshmikantham, and Sivasundaram [5].

Let \( T \in \mathbb{R}, T > 0 \), and let \( T \) be a \( T \)-periodic time scale; in other words, \( T \) is a nonempty closed subset of \( \mathbb{R} \) such that \( t + T \in T \) and \( \mu(t) = \mu(t + T) \) whenever \( t \in T \). Define the delta differential operator \( L \) by

\[
Ly(t) := -(py^\Delta(t)) + q(t)y^\sigma(t), \quad t \in T,
\]

and consider the boundary value problem

\[
Ly(t) = f(t, y^\sigma(t))
\]

with the periodic boundary conditions

\[
y(a) = y(a + T), \quad y^\Delta(a) = y^\Delta(a + T), \quad a \in T,
\]

where

\begin{align*}
(H_1) \quad & p(t) > 0, \quad p(t) = p(t + T), \quad p \text{ is right-dense continuous on } T; \\
(H_2) \quad & q(t) \geq 0, \quad q(t) = q(t + T), \quad q \text{ is right-dense continuous with } \int_a^{a+T} q(\tau) \Delta \tau > 0 \text{ on } T;
\end{align*}

2000 Mathematics Subject Classification. 34B15, 39A10.

Key words and phrases. time scales, boundary value problems, Riccati equation, fixed points.
(H₃) \( f : \mathbb{T} \times [0, \infty) \to [0, \infty) \) is continuous with \( f(t, u) = f(t + T, u) \).

This problem (2), (3) has been studied previously in the discrete case by Atici and Guseinov [6], and by Liu and Ge [7] in the continuous case. Bohner and Peterson [3, Theorem 4.89] mention this problem briefly for general time scales; we will use the approach of [6, 7] rather than [3]. Topal [4, Section 6.3] studied a related periodic problem,

\[-y^\Delta + q(t)y = f(t, y), \quad y^{\rho}(a) = y(b), \quad y^\Delta(\rho(a)) = y^\Delta(b)\]

using the method of upper and lower solutions. Finding solutions to (2) with Sturm-Liouville boundary conditions has also received recent attention; please consult Erbe and Peterson [8, 9, 10] and Erbe, Mathsen, and Peterson [11]. For more on Green’s functions on time scales, see Anderson [12] and Hoffacker [13].

2. Introduction to Time Scales

We mention without proof several foundational results and definitions from the calculus on time scales.

**Theorem 1.** [3, Theorem 1.16 (iv)] If \( t \in \mathbb{T}^\kappa \), and \( f : \mathbb{T} \to \mathbb{R} \) is delta differentiable at \( t \), then

\[
(f^\sigma(t) = f(t) + \mu(t)f^\Delta(t).
\]

**Theorem 2.** [3, Theorem 1.20 (iii)] Assume \( f, g : \mathbb{T} \to \mathbb{R} \) are delta differentiable at \( t \in \mathbb{T}^\kappa \). Then

\[
(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).
\]

**Definition 3.** [3, Definition 2.25] A function \( r : \mathbb{T} \to \mathbb{R} \) is regressive provided

\[
1 + \mu(t)r(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa.
\]

The set of all regressive and rd-continuous functions will be denoted by \( \mathcal{R} \).

**Definition 4.** [3, Theorem 2.30] If \( r \in \mathcal{R} \), then the delta exponential function is given by

\[
e_r(t, s) = \exp\left(\int_s^t r(\tau) \Delta \tau\right) : \mu(\tau) = 0;
\]

\[
e_r(t, s) = \exp\left(\int_s^t \frac{1}{\mu(\tau)} \log(1 + r(\tau)\mu(\tau)) \Delta \tau\right) : \mu(\tau) \neq 0,
\]
where \( \log \) is the principal logarithm.

**Theorem 5.** [\( 3, \) Theorems 2.33, 2.35] Let \( r \in \mathbb{R} \) and fix \( t_0 \in T \). Then \( y = e_r(\cdot, t_0) \) is the solution of the initial value problem

\[
y^\Delta(t) = r(t)y(t), \quad y(t_0) = 1, \quad t \in T^e.
\]

**Lemma 6.** If \( r \in \mathbb{R} \), then \( \ominus r \in \mathbb{R} \), where

\[
\ominus r(t) := \frac{-r(t)}{1 + \mu(t)r(t)}, \quad t \in T^e.
\]

**Theorem 7.** [\( 3, \) Theorem 2.36] If \( r \in \mathbb{R} \), then

1. \( e_r(\sigma(t), s) = (1 + \mu(t)r(t))e_r(t, s) \);
2. \( e_r(t, s) = \frac{1}{e_r(\sigma(t), t)} = e_{\ominus r}(s, t) \);
3. \( e_r(t, \sigma(s)) = \frac{e_r(t, s)}{1 + \mu(t)\sigma(t)} \);
4. \( e_r(t, u)e_r(u, s) = e_r(t, s) \).

**Theorem 8.** [\( 3, \) Theorem 2.77] Let \( r \in \mathbb{R} \), \( a \in T \), and \( y_a \in \mathbb{R} \). The unique solution of the initial value problem

\[
y^\Delta = r(t)y + h(t), \quad y(a) = y_a
\]
is given by

\[
y(t) = e_r(t, a)y_a + \int_a^t e_r(t, \sigma(\tau))h(\tau)\Delta \tau.
\]

### 3. Preliminary Results

In this section we will prove the existence of a periodic solution to a Riccati equation associated with \( Ly = 0 \), and use this Riccati function in the choice of an exponential function that will allow us to factor and integrate the operator equation (1). This in turn will lead to a nice expression for the corresponding Green’s function for \( Ly = 0 \) with boundary conditions (3) in such a way that we can find bounds on it for later use.

**Remark 9.** Let \( a \in T \). We will use the following in the next lemma:

\[
p^* := \max_{t \in [a, a+T]} p(t), \quad q^* := \max_{t \in [a, a+T]} q(t),
\]

\[
z^*(t) := \frac{\mu(t)q^* + \sqrt{\mu^2(t)q^*2 + 4p^*q^*2}}{2}, \quad B := \max_{t \in [a, a+T]} z^*(t).
\]
Theorem 10. Let \( a \in \mathbb{T} \) and assume \((H_1), (H_2)\). The Riccati equation
\[
(6) \quad z^\Delta(t) = q(t) - \frac{z^2(t)}{p(t) + \mu(t)z(t)}, \quad t \in \mathbb{T}
\]
has a unique nonnegative, attracting \(T\)-periodic solution \(z_T\) such that \(0 \leq z_T(t) \leq B\) for all \(t \in \mathbb{T}\) and \(\int_a^{a+T} z_T(t) \Delta t > 0\).

Proof. Clearly \(0 < B\). For \(z_a \geq 0\), let \(z(t; a, z_a)\) be the unique solution of (6) through the point \((a, z_a)\). We show that if \(z_a \in [0, B]\), then \(z(t) \in [0, B]\) for all \(t \in \mathbb{T}\). Let \(z_a \in [0, B]\). If \(z(t; a, z_a) = z(t) = 0\) for some \(t \in [a, a+T]\), there are two cases. If \(t\) is a right-dense point, then \(z(t) = 0\) and \(z^\Delta(t) = q(t) \geq 0\). Therefore \(z\) remains nonnegative. If that \(t\) is a right-scattered point, then \(z^\sigma(t) = \mu(t)q(t) \geq 0\). In either case, \(z\) cannot become negative. Finally, suppose \(z(t; a, z_a) = z(t) > 0\) for some right-scattered \(t \in [a, a+T]\); then
\[
z^\sigma(t) = z(t) + \mu(t)q(t) - \frac{\mu(t)z^2(t)}{p(t) + \mu(t)z(t)} \geq \frac{z(t)p(t)}{p(t) + \mu(t)z(t)} > 0,
\]
so that \(z\) cannot “jump over” the axis and become negative. To verify the upper bound, suppose there exists \(\tau \in \mathbb{T}, \tau \geq \inf\{t \geq a : z^\sigma(t) > z^*(t)\}\) such that \(z^\sigma(\tau) > z^*(\tau)\) and \(z^\Delta(\tau) \geq 0\). If \(\tau\) is a right-scattered point, then \(z(\tau) \leq z^*(\tau)\) and
\[
z^*(\tau) < z^\sigma(\tau) = \mu(\tau)q(\tau) + \frac{z(\tau)p(\tau)}{p(\tau) + \mu(\tau)z(\tau)} \leq \mu(\tau)q^* + \frac{z^*(\tau)p^*}{p^* + \mu(\tau)z^*(\tau)} = z^*(\tau),
\]
a contradiction. If \(\tau\) is a right-dense point, then \(\mu(\tau) = 0\), \(z(\tau) > z^*(\tau)\) and \(z^\Delta(\tau) \geq 0\); thus,
\[
0 \leq z^\Delta(\tau) = q(\tau) - \frac{z^2(\tau)}{p(\tau)} < q^* - \frac{z^2(\tau)}{p^*} = 0,
\]
again a contradiction. Consequently \(0 \leq z(t) \leq B\) for all \(t \in \mathbb{T}\) if \(z_a \in [0, B]\). Note that in particular, \(z(a+T; a, z_a) \in [0, B]\). This motivates us to define the operator \(\mathcal{Y} : [0, B] \to [0, B]\) via \(\mathcal{Y}(z_a) = z(a+T; a, z_a)\) for each \(z_a \in [0, B]\). Since \(\mathcal{Y}\) is a continuous bijection on a compact interval, it has a fixed point \(z^*_a\). Since \(p\) and \(q\) are \(T\)-periodic functions, the unique solution \(z_T := z_T(\cdot; a, z^*_a)\) of the Riccati equation (6) through \((a, z^*_a)\) is nonnegative and \(T\) periodic. To show that \(z_T\) is the unique, attracting, \(T\)-periodic solution, let \(z\) be any other nonnegative solution of (6), and let \(u := z_T - z\). Then
\[
u^\Delta = 0u,
\]
where
\[ \phi := -\frac{p(z + z_T) + \mu z z_T}{(p + \mu z)(p + \mu z_T)}. \]
Note that \( \phi(t) < 0 \) and \( u \) and \( u^\Delta \) are of opposite sign for all \( t \in \mathbb{T} \). Therefore \( u(t) = u(a)e^{\phi(t,a)} \) characterized by Definition 4 and Theorem 5, with \( u(a) = z_T(a) - z(a) \).

Since
\[ 1 + \mu \phi = \frac{p^2}{(p + \mu z)(p + \mu z_T)} > 0, \quad t \in \mathbb{T}, \]
\( e^{\phi(t,a)} > 0 \) for all \( t \in \mathbb{T} \) by [3, Theorem 2.44]. It follows that \( u(t) = z_T(t) - z(t) = u(a)e^{\phi(t,a)} \) is of constant sign on \( \mathbb{T} \), and since \( u \) and \( u^\Delta \) are of opposite sign, \( \lim_{t \to -\infty} u(t) \) exists in \( \mathbb{R} \), and \( \lim_{t \to -\infty} u^\Delta(t) = 0 \) (these limits are taken over \( t \in \mathbb{T} \)). The fact that \( z_T \) and \( p \) are \( T \)-periodic and bounded implies that \( z \) is also bounded, so that \( \phi \) is bounded. Thus \( u(t) \to 0 \) or \( \phi(t) \to 0 \). In either case, \( z(t) \to z_T(t) \).

Consequently, \( z_T \) is the unique, nonnegative, \( T \)-periodic, attracting (asymptotically stable) solution to (6). The fact that \( \int_a^{a+T} z_T(t)\Delta t > 0 \) follows from the assumption that \( q \geq 0 \) is not identically zero on \([a,a + T]\).

**Example 11.** Let \( \mathbb{T} = h\mathbb{Z} \) for any \( h > 0 \). Then \( \mu(t) \equiv h \) and the Riccati equation (6) is equivalent to the first-order difference equation
\[ (7) \quad z(t + h) = \frac{p(t)z(t)}{p(t) + hz(t)} + hq(t), \quad t \in h\mathbb{Z}. \]
If the functions \( p, q \) are taken to be positive constants (functions with period \( h \)), it is easy to verify that the unique positive, period-\( h \) solution of (7) is
\[ z^* = \frac{1}{2} \left[ hq + \sqrt{h^2q^2 + 4pq} \right]. \]
Now let \( z^* \) be the unique \( T \)-periodic solution of (7) for arbitrary \( p, q \) satisfying \((H_1),(H_2)\); since every point is a right-scattered point, \( z^* \) is strictly positive. Then, for any other positive solution \( z \) of (7),
\[ |z(t + h) - z^*(t + h)| = \frac{p^2(t)|z(t) - z^*(t)|}{|p(t) + hz(t)||p(t) + hz^*(t)|} < |z(t) - z^*(t)|. \]
Therefore \( z^* \) is a stable periodic solution as predicted by Theorem 10.

**Corollary 12.** Let \( a \in \mathbb{T} \) and assume \((H_1),(H_2)\). The homogeneous equation \( Lu = 0 \) has a solution \( u \) such that \( u(a) = 1 \) and \( u \) is increasing on \([a, \infty) \cap \mathbb{T}\). Moreover, \( Lu = 0 \) is disconjugate on \([a, \infty) \cap \mathbb{T}\).
Proof. Let \( z \) be the unique solution to (6) found in Theorem 10. Since \( z \geq 0 \) by Theorem 10 and \( p > 0 \) by \((H_1)\), we have \( z/p \in \mathbb{R} \). Consider the function 

\[ u(t) := e_{z/p}(t, a) \]

characterized by Definition 4 and Theorem 5. It is easy to check that \( Lu = 0 \) and \( u(a) = 1 \). Furthermore, \( u'z/p \geq 0 \), so that \( u \) is monotone increasing on \([a, \infty) \cap \mathbb{N}\). The existence of a positive solution on \([a, \infty) \cap \mathbb{N}\) implies that \( Lu = 0 \) is disconjugate; see [3, Theorem 4.53]. \(\square\)

Remark 13. Take \( k \) to be the constant

\[ k := \left[ 1 - e^{z/p}(a + T, a) \right] e^{z/p}(a + T, a) - 1 \]

Note that \( k > 0 \) by Definition 4 since \( p \) is positive, \( \mu \) is nonnegative, and \( z \) is a nontrivial, nonnegative solution of (6) by Theorem 10. If we define

\[ G(t, s) := \int_{t}^{s} \frac{e_{z/p}(t, \sigma(\tau)) e_{z/p}(\sigma(s), \tau)}{k \mu(\tau)} d\tau 
+ \int_{s}^{t+T} \frac{e_{z/p}(t, \sigma(\tau)) e_{z/p}(\sigma(s + T), \tau)}{k \mu(\tau)} d\tau \]

the next lemma will show that \( G(t, s) \) is the Green’s function for \( Ly = 0 \) satisfying boundary conditions (3).

Lemma 14. Assume \((H_1), (H_2)\). Let \( h : \mathbb{T} \to \mathbb{R} \) be right-dense continuous and \( T \) periodic, and let \( z \) be the unique \( T \)-periodic solution of (6). Then the periodic boundary value problem

\[ Ly(t) = h(t), \quad t \in \mathbb{T} \]

with boundary conditions (3) has periodic solution

\[ y(t) = \int_{t}^{t+T} G(t, s) h(s) ds, \]

where \( G(t, s) \) is given by (9).

Proof. Let \( z \) be the unique \( T \)-periodic solution to the Riccati equation found in Theorem 10. We use the monotone solution to the homogeneous problem, \( e_{z/p}(t, a) \) (see the proof to Corollary 12), and consider the following dynamic equation, where for brevity the independent variable \( t \) and the initial time \( a \) have been suppressed:

\[ (py^{\Delta} e_{z/p})^{\Delta} - (zy e_{z/p})^{\Delta} = -h(1 + \mu z/p) e_{z/p}. \]
Using Theorem 2, (4), and Theorem 7 as needed, we expand this out to get

\[(py^\Delta)^{\Delta} e_{z/p} + py^\Delta e_{z/p} z/p - z^{\sigma} (ge_{z/p}^\Delta) - z^\Delta y e_{z/p} = -h(1 + \mu z/p)e_{z/p}\]

\[(py^\Delta)^{\Delta} (1 + \mu z/p) + zy^\Delta - z^{\sigma} y^\Delta (1 + \mu z/p) - z^\Delta y = -h(1 + \mu z/p)\]

\[(py^\Delta)^{\Delta} (1 + \mu z/p) + y^\Delta [z - z^{\sigma} (1 + \mu z/p)]\]

\[+ (-y^{\sigma} + \mu y^\Delta) [z^\Delta + z^{\sigma}/p] = -h(1 + \mu z/p)\]

\[(py^\Delta)^{\Delta} (1 + \mu z/p) - y^\sigma [z(z + \mu z^\Delta)/p + z^\Delta] = -h(1 + \mu z/p)\]

\[(py^\Delta)^{\Delta} \left( \frac{z^2}{p + \mu z} + \Delta \right) y^\sigma = -h,\]

which is \(Ly = h\) since \(z\) is a solution of (6). Integrate (11) from \(t\) to \(t + T\) and use Theorem 7 to get

\[\int_t^{t+T} h(\eta) e_{z/p}(\sigma(\eta), \tau) \Delta\eta.\]

Now since \(z\) and the time scale \(\mathbb{T}\) are \(T\)-periodic, for all \(t \in \mathbb{T}\) we have that

\[e_{z/p}(t + T, t) = e_{z/p}(a + T, a)\]

by Definition 4. This implies that

\[\int_t^{t+T} h(\eta) e_{z/p}(\sigma(\eta), \tau) \Delta\eta.\]

which simplifies, after employing properties of the time-scale exponential, to

\[(12) \quad y^\Delta(t) - z(t) y(t)/p(t) = -\frac{1}{p(t)[e_{z/p}(a + T, a) - 1]} \int_t^{t+T} h(\eta) e_{z/p}(\sigma(\eta), t) \Delta\eta.\]

For computational convenience set

\[(13) \quad \phi(\tau) := -\frac{1}{p(t)[e_{z/p}(a + T, a) - 1]} \int_0^{\tau+T} h(\eta) e_{z/p}(\sigma(\eta), \tau) \Delta\eta.\]

By the variation of constants formula (Theorem 8), (12) has the solution

\[y(t + T) = e_{z/p}(t + T, t) y(t) + \int_t^{t+T} e_{z/p}(t + T, \sigma(\tau)) \phi(\tau) \Delta\tau.\]

Since \(y(t + T) = y(t)\) and \(e_{z/p}(t + T, t) = e_{z/p}(a + T, a)\), we obtain

\[y(t) = \frac{1}{1 - e_{z/p}(a + T, a)} \int_t^{t+T} e_{z/p}(t + T, \sigma(\tau)) \phi(\tau) \Delta\tau.\]

Using Theorem 7, this can be rewritten as

\[y(t) = \frac{-1}{1 - e_{z/p}(a + T, a)} \int_t^{t+T} e_{z/p}(t, \sigma(\tau)) \phi(\tau) \Delta\tau.\]
Thus a solution of \( Lg(t) = h(t), (3) \) is given by

\[
y(t) = \int_t^{t+T} \int_\tau^{t+T} \frac{e_{z/p}(t, \sigma(\tau))e_{z/p}(\sigma(\tau), \tau)h(\eta)}{p(\tau) \left[ 1 - e_{z/p}(a + T, a) \right] \left[ e_{z/p}(a + T, a) - 1 \right]} \Delta \eta \Delta \tau.
\]

Set \( k \) as in (8), and interchange the order of integration to obtain

\[
y(t) = \int_t^{t+T} \left( \int_\tau^{t+T} \frac{e_{z/p}(t, \sigma(\tau))e_{z/p}(\sigma(\tau), \tau)h(\eta)}{k p(\tau)} \Delta \tau \right) h(\eta) \Delta \eta
\]

\[
+ \int_t^{t+2T} \left( \int_\eta^{t+T} \frac{e_{z/p}(t, \sigma(\tau))e_{z/p}(\sigma(\eta), \tau)h(\eta)}{k p(\tau)} \Delta \tau \right) h(\eta) \Delta \eta.
\]

Use the substitutions \( s = \eta \) and \( s = \eta - T \) to arrive at

\[
y(t) = \int_t^{t+T} G(t, s)h(s) \Delta s,
\]

where \( G(t, s) \) is given, as in (9), by

\[
G(t, s) = \int_t^s \frac{e_{z/p}(t, \sigma(\tau))e_{z/p}(\sigma(\tau), \tau)}{k p(\tau)} \Delta \tau + \int_s^{t+T} \frac{e_{z/p}(t, \sigma(\tau))e_{z/p}(\sigma(s + T), \tau)}{k p(\tau)} \Delta \tau
\]

for \( t \leq s \leq t + T \).

\[\square\]

**Lemma 15.** Assume \((H_1), (H_2)\). Let \( h : \mathbb{T} \to \mathbb{R} \) be right-dense continuous, and let \( z, y \) be the solutions of (6), (10) respectively. Then

\[
y(t) \geq \ell \|y\| \quad t \in \mathbb{T}
\]

for

\[
\ell := \int_a^{a+T} \frac{e_{z/p}(a, \sigma(\tau))e_{z/p}(\sigma(a), \tau)}{p(\tau)} \Delta \tau + \int_a^{a+T} \frac{1}{p(\tau)} \Delta \tau.
\]

**Proof.** For the kernel \( G \) given in (9), we have for \( t \leq s \leq t + T \) that

\[
G(t, s) \leq \int_t^{t+T} \frac{e_{z/p}(t, \sigma(\tau))}{k p(\tau)} \left[ e_{z/p}(\sigma(s), \tau) + e_{z/p}(\sigma(s + T), \tau) \right] \Delta \tau
\]

\[
\leq \int_t^{t+T} \frac{1}{k p(\tau)} \left[ e_{z/p}(\sigma(s), t) + e_{z/p}(\sigma(s + T), t) \right] \Delta \tau
\]

\[
\leq \left[ e_{z/p}(\sigma(a + T), a) + e_{z/p}(\sigma(a + 2T), a) \right] \int_a^{a+T} \frac{1}{k p(\tau)} \Delta \tau.
\]
Likewise

\[ G(t, s) = \int_t^s \frac{e_{z/p}(\sigma(s), \tau)}{e_{z/p}(\sigma(\tau), t)} \Delta \tau + \int_s^{t+T} \frac{e_{z/p}(\sigma(s+T), \sigma(s))e_{z/p}(\sigma(s), \tau)}{e_{z/p}(\sigma(\tau), t)} \Delta \tau \]

\[ \geq \int_t^{t+T} \frac{e_{z/p}(t, \sigma(\tau))e_{z/p}(\sigma(s), \tau)}{kp(\tau)} \Delta \tau \]

\[ \geq \int_t^{t+T} \frac{e_{z/p}(t, \sigma(\tau))e_{z/p}(\sigma(t), \tau)}{kp(\tau)} \Delta \tau \]

\[ = \int_a^{a+T} \frac{e_{z/p}(a, \sigma(\tau))e_{z/p}(\sigma(a), \tau)}{kp(\tau)} \Delta \tau. \]

If we define

\[ m := \int_a^{a+T} \frac{e_{z/p}(a, \sigma(\tau))e_{z/p}(\sigma(a), \tau)}{kp(\tau)} \Delta \tau \]

and

\[ M := [e_{z/p}(\sigma(a+T), a) + e_{z/p}(\sigma(a+2T), a)] \int_a^{a+T} \frac{1}{kp(\tau)} \Delta \tau, \]

then

\[ y(t) \geq \frac{m}{M} \|y\| = \ell \|y\|, \quad t \in T. \]

4. **Existence of One or Two Periodic Solutions**

Let \( S \) denote the Banach space \( C_{rd}(\mathbb{T}) \) with the supremum norm

\[ \|y\| = \sup_{t \in \mathbb{T}} |y(t)|. \]

For \( \ell \) as in (15), define the cone \( \mathcal{P} \subset S \) via

\[ \mathcal{P} := \{ y \in S : y(t) \geq \ell \|y\|, t \in T \}. \]

Define the integral operator by

\[ Ay(t) := \int_t^{t+T} G(t, s)f(s, y^\sigma(s)) \Delta s. \]

By Lemma 14 the fixed points of \( A \) are solutions of (2), (3). We first employ the following theorem, due to Krasnoselskii [14].
Theorem 16. Let $S$ be a Banach space, $P \subseteq S$ be a cone, and suppose that $B_1, B_2$ are bounded open balls of $S$ centered at the origin, with $B_1 \subset B_2$. Suppose further that $A : P \cap (B_2 \setminus B_1) \to P$ is a completely continuous operator such that either
\[ \|Au\| \leq \|u\|, \quad u \in P \cap \partial B_1 \quad \text{and} \quad \|Au\| \geq \|u\|, \quad u \in P \cap \partial B_2 \]
or
\[ \|Au\| \geq \|u\|, \quad u \in P \cap \partial B_1 \quad \text{and} \quad \|Au\| \leq \|u\|, \quad u \in P \cap \partial B_2 \]
holds. Then $A$ has a fixed point in $P \cap (B_2 \setminus B_1)$.

Theorem 17. Assume $(H_1) - (H_3)$. Suppose further that there exist positive numbers $0 < r < R < \infty$ such that
\[ (H_4) \quad f(t, y) \leq \frac{y}{MT} \quad \text{for} \quad y \in [0, r], \quad \text{and} \quad f(t, y) \geq \frac{y}{ltmT} \quad \text{for} \quad y \in [R, \infty), t \in T. \]
Then (2), (3) has a positive periodic solution $y$ such that, for $\ell$ as in (15),
\[ \ell r \leq y(t) \leq \frac{R}{\ell}, \quad t \in T. \]

Proof. If $y \in P$, then $Ay(t) \geq \ell \|Ay\|$ for $t \in T$ by Lemma 15, so that $A(P) \subset P$. Moreover, $A$ is completely continuous using standard arguments. Define bounded open balls centered at the origin by
\[ B_1 = \{ y \in S : \|y\| < r \}, \quad B_2 = \{ y \in S : \|y\| < R' \}, \]
where $R' := MR/m$. Then $0 \in B_1 \subset B_2$. For $y \in P \cap \partial B_1$ so that $\|y\| = r$, we have
\[ Ay(t) = \int_t^{t+T} G(t, s)f(s, y^\sigma(s))\Delta s \leq M \int_t^{t+T} f(s, y^\sigma(s))\Delta s \leq \frac{1}{T} \int_t^{t+T} y^\sigma(s)\Delta s \leq \|y\|. \]
Thus, $\|Ay\| \leq \|y\|$ for $y \in P \cap \partial B_1$. Similarly, let $y \in P \cap \partial B_2$, so that $\|y\| = R'$. Then
\[ y^\sigma(t) \geq \ell \|y\| = \frac{m}{M} R' = R, \]
and
\[ Ay(t) \geq m \int_t^{t+T} f(s, y^\sigma(s))\Delta s \geq m \frac{M}{m^2T} \int_t^{t+T} y^\sigma(s)\Delta s \geq \frac{M}{mT} \int_t^{t+T} m \frac{R'}{M} \Delta s = \|y\|. \]
Consequently, \( \|Ay\| \geq \|y\| \) for \( y \in \mathcal{P} \cap \partial B_2 \). By Theorem 16, \( A \) has a fixed point \( y \in \mathcal{P} \cap (\overline{B_{R'}} \setminus B_1) \), which is a positive solution of (2), (3), such that \( r \leq \|y\| \leq R' \). Using the fact that \( y \in \mathcal{P} \) and the definition of \( \ell \) in (15), the bounds on \( y \) follow. \( \square \)

The proof of the next theorem is similar to that just completed.

**Theorem 18.** Assume \((H_1) - (H_3)\). In addition, suppose that there exist positive numbers \( 0 < r < R < \infty \) such that

\[\begin{align*}
(H_5) \quad & f(t,y) \leq \frac{y}{MT} \text{ for } y \in [R, \infty), \text{ and } f(t,y) \geq \frac{y}{\ell m T} \text{ for } y \in [0,r], t \in T.
\end{align*}\]

Then (2), (3) has a positive periodic solution \( y \) such that

\[ \ell r \leq y(t) \leq R/\ell, \quad t \in T. \]

With an additional assumption one can prove the existence of at least two positive periodic solutions to (2), (3). The proofs are modifications of the proof in Theorem 17 and are omitted.

**Theorem 19.** Assume \((H_1) - (H_3)\). Suppose that there exist positive numbers \( 0 < r < N < R < \infty \) such that for \( t \in T \),

\[\begin{align*}
(H_6) \quad & f(t,y) < \frac{N}{MT} \text{ for } y \in [\ell N, N], \text{ and } f(t,y) \geq \frac{y}{\ell m T} \text{ for } y \in [0,r] \cup [R, \infty),
\end{align*}\]

as well. Then (2), (3) has at least two positive periodic solutions \( y_1, y_2 \) such that \( \|y_1\| < N < \|y_2\| \), and

\[ \ell r \leq y_1(t) < N, \quad \ell N < y_2(t) \leq R/\ell, \quad t \in T. \]

**Theorem 20.** Assume \((H_1) - (H_3)\). Additionally, suppose that there exist positive numbers \( 0 < r < N < R < \infty \) such that for \( t \in T \),

\[\begin{align*}
(H_7) \quad & f(t,y) > \frac{N}{\ell m T} \text{ for } y \in [\ell N, N], \text{ and } f(t,y) \leq \frac{y}{MT} \text{ for } y \in [0,r] \cup [R, \infty).
\end{align*}\]

Then (2), (3) has at least two positive periodic solutions \( y_1, y_2 \) such that \( \|y_1\| < N < \|y_2\| \), and

\[ \ell r \leq y_1(t) < N, \quad \ell N < y_2(t) \leq R/\ell, \quad t \in T. \]
Remark 21. Assume \((H_1) - (H_3)\). For \(t \in \mathbb{T}\), define

\[
\begin{align*}
    f_0(t) &:= \lim_{y \to 0^+} \frac{f(t,y)}{y}, \quad f_\infty(t) := \lim_{y \to \infty} \frac{f(t,y)}{y}.
\end{align*}
\]

If \(f_0(t) = 0\) and \(f_\infty(t) = \infty\) for all \(t \in \mathbb{T}\), then \((H_4)\) is satisfied for sufficiently small \(r > 0\) and sufficiently large \(R > 0\). If \(f_0(t) = \infty\) and \(f_\infty(t) = 0\) for all \(t \in \mathbb{T}\), then \((H_5)\) is satisfied. Likewise if \(f_0(t) = f_\infty(t) = \infty\) for all \(t \in \mathbb{T}\), then \((H_6)\) holds, and if \(f_0(t) = f_\infty(t) = 0\) for all \(t \in \mathbb{T}\), then \((H_7)\) holds.

5. Existence of Three Periodic Solutions

In this section we employ the Leggett-Williams Theorem [15] to establish the existence of at least three positive periodic solutions to (2), (3). Before proceeding to the theorem, however, we first introduce some notation.

A map \(\psi\) is a nonnegative continuous concave functional on a cone \(P\) if it satisfies the following conditions:

\[(i) \quad \psi : P \to [0, \infty) \text{ is continuous};\]

\[(ii) \quad \psi(tx + (1 - t)y) \geq t\psi(x) + (1 - t)\psi(y) \quad \text{for all} \quad x, y \in P \text{ and } 0 \leq t \leq 1.\]

Take the same cone \(P\) as before (18), and let

\[
P_c := \{y \in P : \|y\| < c\}
\]

and

\[
P(\psi, b, d) := \{y \in P : b \leq \psi(y), \|y\| \leq d\}.
\]

Theorem 22. Let \(P\) be a cone in the real Banach space \(S\), \(A : \overline{P}_c \to \overline{P}_c\) be completely continuous and \(\psi\) be a nonnegative continuous concave functional on \(P\) with \(\psi(y) \leq \|y\|\) for all \(y \in \overline{P}_c\). Suppose there exists \(0 < \alpha < b < d \leq c\) such that the following conditions hold:

\[(i) \quad \{y \in P(\psi, b, d) : \psi(y) > b\} \neq \emptyset \text{ and } \psi(Ay) > b \text{ for all } y \in P(\psi, b, d);\]

\[(ii) \quad \|Ay\| < \alpha \text{ for } \|y\| \leq \alpha;\]

\[(iii) \quad \psi(Ay) > b \text{ for } y \in P(\psi, b, c) \text{ with } \|Ay\| > d.\]

Then \(A\) has at least three fixed points \(y_1, y_2, \text{ and } y_3\) in \(\overline{P}_c\) satisfying:

\[
\|y_1\| < \alpha, \quad \psi(y_2) > b, \quad \alpha < \|y_3\| \text{ with } \psi(y_3) < b.
\]
Let the nonnegative continuous concave functional $\psi : \mathcal{P} \to [0, \infty)$ by defined by

$$\psi(y) = \min_{t \in [a, a+T]} y(t), \quad y \in \mathcal{P};$$

note that for $y \in \mathcal{P}$, $0 < \psi(y) \leq \|y\|$ by Lemma 15.

**Theorem 23.** Assume $(H_1) - (H_3)$. Suppose that there exist constants $0 < \alpha < b < b/\ell \leq c$ such that, for $t \in T$,

$(H_8)$ $f(t, y) < m\alpha$ if $y \in [0, \alpha]$,

$(H_9)$ $f(t, y) > \frac{b}{MT}$ if $y \in [b, b/\ell]$,

$(H_{10})$ $f(t, y) \leq \frac{c}{MT}$ if $y \in [0, c]$,

where $m$ and $M$ are as defined in (16) and (17), respectively, and $\ell = m/M$ as in (15). Then the boundary value problem (2), (3) has at least three positive periodic solutions $y_1, y_2, y_3$ satisfying

$$\|y_1\| < \alpha, \quad b < \psi(y_2), \quad \|y_3\| > \alpha \quad \text{with} \quad \psi(y_3) < b.$$

**Proof.** Define the operator $A : \mathcal{P} \to \mathcal{S}$ as in (19). As mentioned in the proof to Theorem 17, $A : \mathcal{P} \to \mathcal{P}$ and $A$ is completely continuous. We now show that all of the conditions of Theorem 22 are satisfied. For all $y \in \mathcal{P}$ we have $\psi(y) \leq \|y\|$. If $y \in \mathcal{PC}$, then $\|y\| \leq c$ and assumption $(H_{10})$ implies $f(t, y^\sigma(t)) \leq mc$ for $t \in T$. As a result,

$$\|Ay\| = \max_{t \in T} \int_t^{t+T} G(t, s)f(s, y^\sigma(s))\Delta s \leq \frac{c}{MT} \max_{t \in T} \int_t^{t+T} G(t, s)\Delta s \leq \frac{c}{MT} \max_{t \in T} \int_t^{t+T} M\Delta s = c.$$

Therefore $A : \mathcal{PC} \to \mathcal{PC}$. In the same way, if $y \in \mathcal{P}_\alpha$, then assumption $(H_8)$ yields $f(t, y^\sigma(t)) < m\alpha$ for $t \in T$; as in the argument above, it follows that $A : \mathcal{PC}_\alpha \to \mathcal{P}_\alpha$. Hence, condition $(ii)$ of Theorem 22 is satisfied. To check condition $(i)$ of Theorem 22, choose $y^p(t) \equiv b/\ell$ for $t \in T$. Then $y^p \in \mathcal{P}(\psi, b, b/\ell)$ and $\psi(y^p) = \psi(b/\ell) > b$, so that $\{y \in \mathcal{P}(\psi, b, b/\ell) : \psi(y) > b\} \neq \emptyset$. Consequently, if $y \in \mathcal{P}(\psi, b, b/\ell)$, then $b \leq y^\sigma(s) \leq b/\ell$ for $s \in T$. From assumption $(H_9)$ we have that

$$f(t, y^\sigma(t)) > \frac{b}{mT}.$$
for all \( t \in \mathbb{T} \); we see that
\[
\psi(Ay) = \min_{t \in [a, a+T]} \int_t^{a+T} G(t, s) f(s, y^{\sigma}(s)) \Delta s \geq m \int_a^{a+T} f(s, y^{\sigma}(s)) \Delta s
\]
> \( m \int_a^{a+T} \frac{b}{mT} \Delta s = b \).

Thus we have
\[
\psi(Ay) > b, \quad y \in \mathcal{P}(\psi, b, b/\ell),
\]
so that condition (i) of Theorem 22 holds. Lastly we consider Theorem 22 (iii).

Suppose \( y \in \mathcal{P}(\psi, b, c) \) with \( \|Ay\| > b/\ell \). By the definitions of \( \psi \) and the cone \( \mathcal{P} \),
\[
\psi(Ay) = \min_{t \in \mathbb{T}} Ay(t) \geq \ell \|Ay\| > \ell b/\ell = b.
\]
An application of Theorem 22 yields the conclusion.

\[\Box\]

6. Further Considerations

There are several related second-order equations that may be accessible to the methods used in the previous sections. We could just as well study the second-order nabla problem
\[
- (py^\nabla)(t) + q(t)y^\rho(t) = f(t, y^\rho(t)), \quad t \in \mathbb{T}.
\]
In particular, the discussion would be facilitated by the factored equation corresponding to (11) given by
\[
- (py^\nabla \hat{e}_{\frac{\nu}{p}}) \nabla + (\hat{z}y \hat{e}_{\frac{\nu}{p}}) \nabla = h(1 - \nu \hat{z}/p) \hat{e}_{\frac{\nu}{p}},
\]
where \( \hat{e}_r(\cdot, a) \) is the nabla exponential function, and \( \hat{z} \) is the solution to the nabla Riccati equation
\[
\hat{z}^\nabla(t) = q(t) - \frac{\hat{z}^2(t)}{p(t) - \nu(t) \hat{z}(t)}.
\]
Furthermore, we might consider the second-order delta nabla equation
\[
(22) - (py^\Delta)(t) + q(t)y(t) = h(t), \quad t \in \mathbb{T}.
\]
This time the factored form of (22) is
\[
(23) - (py^\Delta \hat{e}_{\frac{\nu}{p}}) \nabla + (zy \hat{e}_{\frac{\nu}{p}}) \nabla = h \hat{e}_{\frac{\nu}{p}},
\]
where this \( z \) solves a modified nabla Riccati equation,
\[
z^\nabla(t) = q(t) - \frac{(\hat{z}^\rho(t))^2}{p^\rho(t) + \nu(t) \hat{z}^\rho(t)}.
\]
To show this we would expand (23) using the product rule for nabla derivatives, the fact that a solution satisfies $y^{\Delta \rho} = y^{\nabla}$, and the equivalence
\[
e_{\frac{z}{p}}(t, a) = e_{\frac{\sigma}{\sigma}}(t, a) = e_{\Delta}(t, a) = e_{\Delta}(t, a)
\]
according to Messer [4, Theorem 4.17]. In the same way, the nabla delta equation
\[(24) \quad - (py^{\nabla})^{\Delta} + q(t)y(t) = h(t), \quad t \in \mathbb{T}
\]
has the corresponding form
\[
-(py^{\nabla}e_{\frac{z}{p}})^{\Delta} + (zy^{\nabla}e_{\frac{z}{p}})^{\Delta} = h e_{\frac{z}{p}}
\]
for a solution $z$ of
\[
z^{\Delta}(t) = q(t) - \frac{(z^{\sigma}(t))^{2}}{p^{\sigma}(t) - \mu(t)z^{\sigma}(t)}.
\]
where
\[
e_{\frac{z}{p}}(t, a) = e_{\frac{\sigma}{\sigma}}(t, a) = e_{\Delta}(t, a).
\]

REFERENCES


Department of Mathematics and Computer Science, Concordia College, Moorhead, MN 56562 USA

E-mail address: andersod@cord.edu