AN EVEN-ORDER THREE-POINT BOUNDARY VALUE PROBLEM ON TIME SCALES

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ABSTRACT. We study the even-order dynamic equation $(-1)^n x^{(\Delta \nabla)^n}(t) = \lambda h(t) f(x(t)), \ t \in [a, c]$ satisfying the boundary conditions $x^{(\Delta \nabla)^i}(a) = 0$ and $x^{(\Delta \nabla)^i}(c) = \beta x^{(\Delta \nabla)^i}(b)$ for $0 \leq i \leq n - 1$. The three points $a, b, c$ are from a time scale $\mathbb{T}$, where $0 < \beta(b-a) < c-a$ for $b \in (a, c)$, $\beta > 0$, $f$ is a positive function, and $h$ is a nonnegative function that is allowed to vanish on some subintervals of $[a, c]$ of the time scale.

1. INTRODUCTION

In this paper we are concerned with the even-order dynamic equation
\begin{equation}
(-1)^n x^{(\Delta \nabla)^n}(t) = \lambda h(t) f(x(t)), \ t \in [a, c], \ n \in \mathbb{N},
\end{equation}
satisfying the three-point boundary conditions
\begin{equation}
x^{(\Delta \nabla)^i}(a) = 0, \quad x^{(\Delta \nabla)^i}(c) = \beta x^{(\Delta \nabla)^i}(b)
\end{equation}
for $0 \leq i \leq n - 1$, where, for $\beta > 0$,
\begin{equation}
d := c - a - \beta(b-a) > 0;
\end{equation}
to avoid overlap in the boundary conditions, we take $a, b, c \in \mathbb{T}$ such that $\sigma^{n-1}(a) < \rho^{n-1}(b)$ and $\sigma^{n-1}(b) < \rho^{n-1}(c)$. Here $\lambda > 0$, $f$ is a positive left-dense continuous function, and $h$ is a nonnegative left-dense continuous function that is allowed to vanish on some subintervals of $[a, c]$ of the time scale (see the condition in (14)). A solution to (1), (2) is defined on $[\rho^n(a), \sigma^n(c)]$.

This problem is an extension of the second-order, three-point problem studied by Ma [9, 10, 11] on the unit interval, and in [1] on time scales with a delta-nabla differential operator. Please see [2, 3], [4], and [5] for more problems with delta-nabla derivatives, and the recent books by Bohner and Peterson [7, 8] for more on general time scales. Our results are new in the continuous case, the discrete case, and the general time scales case. For example, if $n = 2$ and $\mathbb{T} = \mathbb{Z}$, then (1) becomes the central difference equation
\begin{equation}
x(t+2) - 4x(t+1) + 6x(t) - 4x(t-1) + x(t-2) = \lambda h(t) f(x(t)),
\end{equation}
with boundary conditions
\begin{equation}
x(a) = 0, \quad x(a+1) - 2x(a) + x(a-1) = 0, \quad x(c) = \beta x(b),
\end{equation}
\begin{equation}
x(c+1) - 2x(c) + x(c-1) = \beta[x(b+1) - 2x(b) + x(b-1)]
\end{equation}
for integers $a, b, c$.

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2. Green's function

Shortly we will be concerned with a completely continuous operator whose kernel is the Green’s function for the homogeneous problem
\[-x^{\Delta \nabla} (t) = 0, \quad t \in [a, c]\]
satisfying boundary conditions (2). First, we need a few results on the related second-order homogeneous problem
\[-x^{\Delta \nabla} (t) = 0, \quad t \in [a, c],\]
(4)

\[x(a) = 0, \quad x(c) = \beta x(b).\]
(5)

Lemma 1. The homogeneous boundary value problem (4), (5) has only the trivial solution if and only if \(d \neq 0\) for \(d\) given in (3).

Proof. A general solution of (4) is \(x(t) = pt + q\). An application of the boundary conditions in (5) yields the equations \(q = -ap\) and \(pd = 0\). If \(d \neq 0\), then \(p = 0\) and \(q = 0\), so that \(x\) is trivially zero. \(\square\)

Theorem 2. Let \(G(t,s)\) be Green’s function for the boundary value problem (4), (5). Then
\[G(t,s) = \begin{cases} \frac{1}{d}[(c-s) - \beta(b-s)] : t \leq s \\
\frac{1}{d}[(c-t) - \beta(b-t)] : s \leq t \end{cases} \] for \(s \in [a, b]\)

for \(s \in [b, c]\)

G(t,s) = \begin{cases} \frac{1}{d}(t-a)(c-s) : t \leq s \\
\frac{1}{d}[(s-a)(c-t) + \beta(b-a)(t-s)] : s \leq t \end{cases}.

Proof. From Green’s function theory,

where \(v_i(t, s) = u_i(t, s) - t + s\) for \(i = 1, 2\). We proceed on the branches of the Green’s function.

Let \(s \in [a, b]\). Then \(u_1\) satisfies

\[-u_1^{\Delta \nabla} = 0, \quad u_1(a, s) = 0,\]

while \(v_1\) satisfies

\[v_1(c, s) = \beta v_1(b, s).\]

It follows that

\[u_1(t, s) = (t-a)p(s), \quad v_1(t, s) = (t-a)p(s) - t + s\]

for some function \(p\). Applying the conditions above and solving for \(p\), we have

\[p(s) = \frac{1}{d}[(c-s) - \beta(b-s)]\]

for \(d\) as in (3).
Now let $s \in [b, c]$. In this case $u_2$ likewise satisfies
\[-u_2^{\Delta \nabla} \equiv 0, \quad u_2(a, s) = 0,
\]
but
\[v_2(c, s) = \beta u_1(b, s).
\]
Consequently
\[u_2(t, s) = (t - a)p(s), \quad v_2(t, s) = (t - a)p(s) - t + s
\]
and the conditions above result in
\[p(s) = \frac{1}{d}(c - s).
\]
Some algebraic rearranging give the expressions in (6).

\[\square\]

**Lemma 3.** Let $d > 0$. Then the Green’s function $G(t, s)$ in (6) satisfies
\[G(t, s) \geq \left(\frac{t - a}{c - a}\right) G(c, s), \quad t, s \in [a, c].
\]

**Proof.** Note that $G(t, s)$ may also be written as
\[
G(t, s) = \begin{cases} 
  s \in [a, b] & : \begin{cases} 
    t - a + \frac{1}{d}(t - a)(s - a)(\beta - 1) : t \leq s \\
    s - a + \frac{1}{d}(s - a)(t - a)(\beta - 1) : s \leq t 
  
end{cases} \\
  s \in [b, c] & : \begin{cases} 
    \frac{1}{d}(t - a)(c - s) : t \leq s \\
    s - a + \frac{t - a}{d} [\beta(b - a) - (s - a)] : s \leq t. 
  
end{cases}
\end{cases}
\]
We proceed sequentially on the branches of this Green’s function.

(i) Fix $s \in [a, b]$ and $t \leq s$. Then
\[G(t, s) = t - a + \frac{1}{d}(t - a)(s - a)(\beta - 1)
\]
and
\[\left(\frac{t - a}{c - a}\right) G(c, s) = \frac{(t - a)(s - a)}{c - a} + \frac{1}{d}(t - a)(s - a)(\beta - 1).
\]
Since $t - a > (t - a)(s - a)/(c - a)$, the result follows.

(ii) Let $s \in [a, b]$ and $t \geq s$. Then
\[G(t, s) = s - a + \frac{1}{d}(s - a)(t - a)(\beta - 1)
\]
and
\[\left(\frac{t - a}{c - a}\right) G(c, s) = \frac{(s - a)(t - a)}{c - a} + \frac{1}{d}(s - a)(t - a)(\beta - 1).
\]
Since $s - a > (t - a)(s - a)/(c - a)$, the result holds.

(iii) Take $s \in [b, c]$ and $t \leq s$. Then
\[G(t, s) = \frac{1}{d}(t - a)(c - s)
\]
and
\[\left(\frac{t - a}{c - a}\right) G(c, s) = \frac{(t - a)(c - s)}{d(c - a)} \beta(b - a).
\]
Since $d > 0$, $1 > \beta(b-a)/(c-a)$ and the result follows.

(iv) For $s \in [b, c]$ and $t \geq s$,

$$G(t, s) = s - a + \frac{1}{d}(t - a)[\beta(b - a) - (s - a)]$$

and

$$\left(\frac{t - a}{c - a}\right)G(c, s) = \frac{(s - a)(t - a)}{c - a} + \frac{(t - a)}{d}[\beta(b - a) - (s - a)].$$

Since $s - a > (t - a)(s - a)/(c - a)$, the theorem is proven. \hfill \Box

**Lemma 4.** Let $d > 0$. Then the Green's function $G(t, s)$ in (6) satisfies

$$G(t, s) > 0, \quad t, s \in (a, c).$$

**Proof.** By Lemma 3, it suffices to show that $G(c, s) > 0$ for $s \in (a, c)$. For $s \in (a, b]$,

$$G(c, s) = \frac{\beta}{d}(s - a)(c - b) > 0,$$

and for $s \in [b, c)$,

$$G(c, s) = \frac{\beta}{d}(b - a)(c - s) > 0.$$

\hfill \Box

**Lemma 5.** Let $d > 0$. Then the Green's function $G(t, s)$ in (6) satisfies

$$G(t, s) \leq G(s, s), \quad t, s \in [a, c], \quad 0 < \beta \leq 1,$$

and

$$G(t, s) \leq \max\{G(s, s), G(c, s)\}, \quad t, s \in [a, c], \quad 1 < \beta < \frac{c - a}{b - a}.$$  

**Proof.** We again deal with the branches of the Green's function.

(i) Fix $s \in [a, c]$ and consider any $t$ with $a \leq t \leq s$. Then $G(a, s) = 0$ and $G(t, s)$ is increasing in $t$ for all $t \in (a, s]$, for any $\beta \in (0, \frac{c - a}{b - a})$. Therefore $G(t, s) \leq G(s, s)$.

(ii) Let $s \in [a, b]$ and take $s \leq t \leq c$. Here $G(t, s)$ is decreasing in $t$ if $0 < \beta \leq 1$, so that $G(t, s) \leq G(s, s)$. If $1 < \beta < \frac{c - a}{b - a}$, however, the function is increasing in $t$ and $G(t, s) \leq G(c, s)$.

(iii) Take $s \in [b, c]$, $s \leq t \leq c$, and $\beta \in (0, 1]$. Since $b \leq s$ and $\beta \leq 1$, $G(t, s)$ is nonincreasing in $t$, so that $G(t, s) \leq G(s, s)$.

(iv) For $s \in [b, c]$, $s \leq t \leq c$, and $\beta \in (1, \frac{c - a}{b - a})$, our analysis depends on the placement of $s$. If $s \in \{b, \beta(b - a) + a\}$, then $G(t, s)$ is nondecreasing in $t$ and $G(t, s) \leq G(c, s)$. Otherwise, for $s \in (\beta(b - a) + a, c]$, $G(t, s)$ is nonincreasing in $t$ and $G(t, s) \leq G(s, s)$. \hfill \Box

**Lemma 6.** Let $d > 0$. For fixed $s \in [a, c]$, the Green's function $G(t, s)$ in (6) satisfies

$$\inf_{t \in [b, c]} G(t, s) \geq m\|G(\cdot, s)\|,$$

where

$$m := \min\left\{\frac{\beta(c - b)}{d}, \frac{\beta(b - a)}{c - a}, \frac{b - a}{c - a}\right\} > 0$$

and $\| \cdot \|$ is the supremum norm, $\|x\| = \sup\{|x(t)| : t \in [a, c]\}$. 

**Proof.** First consider the case where $0 < \beta \leq 1$; from Lemma 5, $\|G(\cdot, s)\| = G(s, s)$. By the second boundary condition we know that $G(b, s) \geq G(c, s)$, so that
\[
\min_{t \in [b, c]} G(t, s) = G(c, s).
\]
For $s \in [a, b]$ we have from the branches in (6) that
\[
G(c, s) \geq \frac{\beta(c - b)}{d} G(s, s),
\]
and for $s \in [b, c]$ we have
\[
G(c, s) \geq \frac{\beta(b - a)}{c - a} G(s, s).
\]
Next consider the case $1 < \beta < \frac{c - a}{b - a}$. The second boundary condition this time implies
\[
\min_{t \in [b, c]} G(t, s) = G(b, s);
\]
using Lemma 5 we have
\[
\|G(\cdot, s)\| = \max\{G(c, s), G(s, s)\}.
\]
By using (6) and the cases in the proof of Lemma 5, we see that
\[
G(b, s) \geq \frac{b - a}{c - a} G(c, s)
\]
for $s \in [a, \beta(b - a) + a)$, and
\[
G(b, s) \geq \frac{b - a}{c - a} G(s, s)
\]
for $s \in [\beta(b - a) + a, c]$.

**Theorem 7.** Let $d > 0$. For $G$ as in (6), take $H_1(t, s) := G(t, s)$, and recursively define
\[
H_j(t, s) = \int_a^c H_{j-1}(t, w) G(w, s) \nabla w
\]
for $2 \leq j \leq n$. Then $H_n(t, s)$ is Green’s function for the homogeneous problem
\[
(-1)^n x^{(\Delta \nabla)^n} (t) = 0, \quad t \in [a, c]
\]
satisfying boundary conditions (2).

**Lemma 8.** Let $d > 0$. The Green’s function $H_n(t, s)$ in Theorem 7 satisfies
\[
0 \leq H_n(t, s) \leq L^n \|G(\cdot, s)\|, \quad t, s \in [a, c]
\]
and
\[
H_n(t, s) \geq m^n K^{n-1} \|G(\cdot, s)\|, \quad t \in [b, c], \quad s \in [a, c],
\]
where $m$ is given in (8) and
\[
L := \int_a^c \|G(\cdot, w)\| \nabla w > 0, \quad K := \int_b^c \|G(\cdot, w)\| \nabla w > 0.
\]

**Proof.** Use induction on $n$ and Lemma 6.
Example 9. Let \( n = 2, T = \mathbb{R}, a = 0, c = 1, \) and \( \beta \in (0,1] \). Then the Green’s function in Theorem 7 is

\[
H_2(t, s) = \begin{cases} 
   s \in [0, b] & : t \leq s \\
   s \in [b, 1] & : s \leq t 
\end{cases}
\]

\[
H_2(t, s) = \begin{cases} 
   -\frac{t^3}{6d} (d - s + \beta s) + p_1(s) t & : t \leq s \\
   -\frac{t^3}{6d} (d - s + \beta s) + p_1(s) t + \frac{1}{6} (t - s)^3 & : s \leq t 
\end{cases}
\]

where

\[
p_1(s) := \frac{1}{6d} \left[ -(1 - s)^3 + \frac{(1 - \beta b^3)(d - s + \beta s)}{d} + \beta (b - s)^3 \right]
\]

and

\[
p_2(s) := \frac{1}{6d} \left[ -(1 - s)^3 + \frac{(1 - \beta b^3)(1 - s)}{d} \right].
\]

By Lemma 8, \( H_2(t, s) \) satisfies

\[
0 \leq H_2(t, s) \leq LG(s, s), \quad t, s \in [0, 1]
\]

and

\[
H_2(t, s) \geq m^2 LG(s, s), \quad t \in [b, 1], \quad s \in [0, 1],
\]

where \( m = \min \left\{ \frac{\beta(1-b)}{d}, \beta b \right\}, \ K = \int_b^1 G(s, s) ds, \) and \( L = \int_0^1 G(s, s) ds = \frac{1 - \beta b^3}{6d}. \)

3. Cone Compression and Expansion of Functional Type

In this section, we provide some background material from the theory of cones in Banach spaces.

Definition 10. Let \( E \) be a real Banach space. A nonempty, closed, convex set \( P \subset E \) is called a cone, if it satisfies the following two conditions:

(i) \( x \in P, \lambda \geq 0 \) implies \( \lambda x \in P; \)
(ii) \( x \in P, -x \in P \) implies \( x = 0. \)

Every cone \( P \subset E \) induces an ordering in \( E \) given by

\[
x \leq y \quad \text{if and only if} \quad y - x \in P.
\]

Definition 11. An operator is called completely continuous, if it is continuous and maps bounded sets into precompact sets.

Definition 12. A map \( \alpha \) is said to be a nonnegative continuous functional on a cone \( P \) of a real Banach space \( E \) if

\[
\alpha : P \to [0, \infty)
\]

is continuous.
Let $\alpha$ and $\gamma$ be nonnegative continuous functionals on $P$; then, for positive real numbers $r$ and $R$, we define the following sets:

\[(12)\quad P(\gamma, R) = \{x \in P : \gamma(x) < R\}\]

and

\[(13)\quad P(\gamma, \alpha, r, R) = \{x \in P : r < \alpha(x) \text{ and } \gamma(x) < R\}.
\]

We now state the functional-type cone expansion-compression fixed point theorem [6].

**Theorem 13.** Let $P$ be a cone in a real Banach space $E$, and let $\alpha$ and $\gamma$ be nonnegative continuous functionals on $P$. Assume $P(\gamma, \alpha, r, R)$ as in (13) is a nonempty bounded subset of $P$, $P(\gamma, R) \subseteq P(\gamma, \alpha, r, R)$ for these sets as in (12). If one of the two conditions

(H1) $\alpha(Ax) \leq r$ for all $x \in \partial P(\alpha, r)$, $\gamma(Ax) \geq R$ for all $x \in \partial P(\gamma, R)$, $\inf_{x \in \partial P(\gamma, R)} \|Ax\| > 0$, and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \geq 1$, and $\mu \in (0, 1)$ the functionals satisfy the properties

$\alpha(\lambda y) \geq \lambda \alpha(y)$, $\gamma(\mu z) \leq \mu \gamma(z)$, and $\alpha(0) = 0$

or

(H2) $\alpha(Ax) \geq r$ for all $x \in \partial P(\alpha, r)$, $\gamma(Ax) \leq R$ for all $x \in \partial P(\gamma, R)$, $\inf_{x \in \partial P(\alpha, r)} \|Ax\| > 0$, and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0, 1]$ and $\mu \geq 1$ the functionals satisfy the properties

$\alpha(\lambda y) \leq \lambda \alpha(y)$, $\gamma(\mu z) \geq \mu \gamma(z)$, and $\gamma(0) = 0$

is satisfied, then $A$ has at least one positive fixed point $x^*$ such that

$r \leq \alpha(x^*)$ and $\gamma(x^*) \leq R$.

4. **Main Result**

Let $E = C_{ld}[a, c]$ be the Banach space with the sup-norm, $\|x\| = \sup\{|x(s)| : a \leq s \leq c\}$, and define the cone $P \subset E$ by

$P = \{x \in E : x(t) \geq 0 \text{ for all } t \in [a, c]\}$.

Define the nonnegative continuous functionals $\alpha$ and $\gamma$ on $P$ by

$\gamma(x) := \min_{t \in [b, c]} x(t)$

and

$\alpha(x) := \|x\|$. Pick any nonnegative left-dense continuous function $h$ defined on $[a, c]$, such that the constant

\[(14)\quad h_b \equiv \int_b^c G(c, s) h(s) \nabla s > 0;\]
moreover, define another constant

\begin{equation}
    h_a \equiv \int_a^c \max\{G(s, s), G(c, s)\} \, h(s) \, \nabla s.
\end{equation}

From Theorem 7, it follows that \( x \) is a solution of (1), (2) if and only if

\[ x(t) = \lambda \int_a^c H_n(t, s) h(s) f(x(s)) \, \nabla s \]

for a given \( \lambda \). Therefore, define the operator

\[ T x(t) = \lambda \int_a^c H_n(t, s) h(s) f(x(s)) \, \nabla s; \]

we look for fixed points of this operator in terms of \( h_a, h_b \), and conditions on \( f \).

**Theorem 14.** Let \( d > 0 \), and let \( h : [a, c] \to [0, \infty) \) be a left-dense continuous function satisfying (14). Suppose there exist positive real numbers \( r \) and \( R \) with \( r < R \) and a left-dense continuous function \( f : [0, \infty) \to [0, \infty) \) such that the following conditions are met:

\begin{enumerate}[(i)]
    \item \( f(u) \leq \frac{r}{\lambda L^n - 1 h_a} \) for \( u \in [0, r] \),
    \item \( f(u) \geq \frac{R}{\lambda m^n K^n - 1 h_b} \) for \( u \in [R, \infty) \).
\end{enumerate}

Then the operator \( T \) has at least one fixed point \( x^* \) such that

\[ r \leq \alpha(x^*) \quad \text{and} \quad \gamma(x^*) \leq R. \]

**Proof.** We will invoke the fixed point theorem of cone compression and expansion of functional type (Theorem 13) once we have shown that the hypotheses \((H1)\) have been satisfied. We have that \( P \) is a cone in the Banach space \( E \) with the sup norm. For all \( u \in P \), \( Tu \) is nonnegative and left-dense continuous, thus

\[ T : P \to P. \]

For \( z \in \partial P(\gamma, R) \) and \( 0 \leq \nu \leq 1 \),

\[ \gamma(\nu z) = \min_{t \in [b, c]} \nu z(t) = \nu \min_{t \in [b, c]} z(t) = \nu \gamma(z) \]

and similarly for \( y \in \partial P(\gamma, R) \) and \( \lambda \geq 1 \),

\[ \alpha(\lambda y) = \|\lambda y\| = \lambda \|y\| = \lambda \alpha(y). \]

If \( x \in \overline{P(\alpha, r)} \), then

\[ \gamma(x) = \min_{t \in [b, c]} x(t) \leq \|x\| \leq r < R, \]

so that \( x \in P(\gamma, R) \) and \( \overline{P(\alpha, r)} \subseteq P(\gamma, R) \).

If \( x \in \partial P(\alpha, r) \), then

\[ x(s) \leq r \]
for \( s \in [a, c] \); thus, from (9) and condition (i),
\[
T x(t) = \lambda \int_a^c H_n(t, s) h(s) f(x(s)) \nabla s
\]
\[
\leq \lambda L_{n-1} \int_a^c \| G(\cdot, s) \| h(s) f(x(s)) \nabla s
\]
\[
\leq \lambda L_{n-1} \left( \frac{r}{\lambda L_{n-1} h_a} \right) \int_a^c \| G(\cdot, s) \| h(s) \nabla s
\]
\[
= \frac{r}{h_a} \int_a^c \max \{ G(c, s), G(s, s) \} h(s) \nabla s
\]
\[
= r,
\]
for any \( t \in [a, c] \). Therefore, \( \alpha(Tx) \leq r \).

If \( x \in \partial P(\gamma, R) \), then
\[
x(s) \geq R
\]
for \( s \in [b, c] \); using (10) and condition (ii),
\[
\gamma(Tx) = \min_{t \in [b, c]} \lambda \int_a^c H_n(t, s) h(s) f(x(s)) \nabla s
\]
\[
\geq \lambda m^n K^{n-1} \int_a^c \| G(\cdot, s) \| h(s) f(x(s)) \nabla s
\]
\[
\geq \lambda m^n K^{n-1} \int_b^c \| G(\cdot, s) \| h(s) f(x(s)) \nabla s
\]
\[
\geq \frac{R}{h_b} \int_b^c G(c, s) h(s) \nabla s
\]
\[
= R.
\]
It follows that \( \gamma(Tx) \geq R \), and
\[
\|Tx\| \geq \gamma(Tx) \geq R > 0;
\]
consequently,
\[
\inf_{x \in \partial P(\gamma, R)} \|Tx\| \geq R > 0.
\]

Therefore the hypotheses of the fixed point theorem of cone compression and expansion of functional type are satisfied. Thus the operator \( T \) has a fixed point \( x^* \) such that
\[
r \leq \alpha(x^*) \quad \text{and} \quad \gamma(x^*) \leq R.
\]

5. Eigenvalue interval

By conditions (i) and (ii) of Theorem 14, the boundary value problem (1), (2) has a solution if, given \( \lambda > 0 \), there exists an ld-continuous, nonnegative function \( f \) and real numbers \( 0 < r < R \) such that
\[
0 < \lambda \leq \frac{r}{L_{n-1} h_a f(u)}, \quad u \in [0, r],
\]
and
\[ \frac{R}{m^n K^{n-1} h_b f(u)} \leq \lambda, \quad u \in [R, \infty). \]

Clearly these inequalities follow if
\[ 0 < \lambda \leq \frac{u}{l^{n-1} h_a f(u)}, \quad u \in [0, r], \]
and
\[ \frac{u}{m^n K^{n-1} h_b f(u)} \leq \lambda, \quad u \in [R, \infty). \]

Consequently, set
\[ f_0 := \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_{\infty} := \lim_{u \to \infty} \frac{f(u)}{u}; \]
we have the following result.

**Theorem 15.** Suppose \( d > 0 \). Then for each \( \lambda \) satisfying
\[ \frac{1}{m^n K^{n-1} h_{\infty} f_{\infty}} < \lambda < \frac{1}{L^{n-1} h_a f_0} \]
there exists at least one positive solution of (1), (2) for \( m, L, h_b, \) and \( h_a \) as in (8), (11), (14), and (15), respectively.

**Corollary 16.** Suppose \( d > 0 \). If \( f \) is superlinear (i.e., \( f_0 = 0 \) and \( f_{\infty} = \infty \)), then for any \( \lambda > 0 \) the boundary value problem (1), (2) has at least one positive solution.

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