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GREEN'S FUNCTION FOR A THIRD-ORDER GENERALIZED RIGHT FOCAL PROBLEM

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Abstract. We determine the Green's function for the third-order three-point generalized right focal boundary value problem

$$x'''(t) = 0 \quad t_1 \le t \le t_3$$
$$x(t_1) = x'(t_2) = 0$$
$$\gamma x(t_3) + \delta x''(t_3) = 0,$$

$\delta > 0$, and determine conditions on the coefficients and boundary points to ensure its positivity. Then we use the Krasnoselskii and Leggett and Williams fixed point theorems to prove the existence of solutions to the nonlinear problem

$$x'''(t) = f(t, x(t)) \quad t_1 \le t \le t_3$$
$$x(t_1) = x'(t_2) = 0$$
$$\gamma x(t_3) + \delta x''(t_3) = 0,$$

for certain constraints placed on nonnegative f.

1. Finding Green's function

We are concerned with the third-order three-point generalized right focal boundary value problem

$$x'''(t) = 0, \quad t_1 \le t \le t_3 \tag{1.1}$$

$$x(t_1) = x'(t_2) = 0 \tag{1.2}$$
$$\gamma x(t_3) + \delta x''(t_3) = 0.$$

Here we assume

(i) $\gamma \ge 0$ and $\delta > 0$;
(ii) $k := 2\delta + \gamma(t_3 - t_1)(t_3 - 2t_2 + t_1) > 0$;
(iii) $t_1 < t_2 < t_3$ are real numbers with

$$t_2 - t_1 > t_3 - t_2;$$

see Lemma 1.1 and Theorem 2.1.

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This is a generalization of the $(3,3)$ right focal Green's function on $\mathbb{R}$ related to that in [5] and used in [6, 8], where $\delta = 1$ and $\gamma = 0$. For more on focal problems and related topics, see the books by Agarwal [1] and Agarwal, Wong, and O'Regan [2]; see also [7]. For other papers on the existence of positive solutions, see Agarwal, Bohner, and Wong [3], Agarwal and O'Regan [4], Davis, Erbe, and Henderson [13], and Erbe and Wang [14]. For more on the use of the Leggett-Williams theorem, see Avery's work [9, 10, 11] and Avery and Henderson [12]. This list is by no means exhaustive; hopefully it will grant the reader a flavor of the techniques used and the problems considered.

Lemma 1.1. *The number k satisfies*

$$k = 2\delta + \gamma(t_3 - t_1)(t_3 - 2t_2 + t_1) \neq 0 \tag{1.3}$$

if and only if the boundary value problem (1.1), (1.2) *has only the trivial solution.*

Proof. A general solution of (1.1) is $x(t) = at^2 + bt + c$. The boundary conditions at t_1, t_2, and t_3 lead to the three equations

$$\begin{aligned} at_1^2 + bt_1 + c &= 0 \\ 2at_2 + b &= 0 \\ \gamma(at_3^2 + bt_3 + c) + 2\delta a &= 0. \end{aligned}$$

The determinant of the coefficients for this system is k. It follows that $a = b = c = 0$ if and only if $k \neq 0$. This implies the given boundary value problem (1.1), (1.2) has only the trivial solution if and only if $k \neq 0$. □

Theorem 1.1. *Assume for k given in* (1.3) *that $k > 0$. Then the Green's function for the homogeneous problem* (1.1) *satisfying the boundary conditions* (1.2) *is given via*

$$G(t,s) = \begin{cases} s \in [t_1, t_2] & : \begin{cases} u_1(t,s) & : t \leq s \\ v_1(t,s) & : t \geq s \end{cases} \\ \\ s \in [t_2, t_3] & : \begin{cases} u_2(t,s) & : t \leq s \\ v_2(t,s) & : t \geq s \end{cases} \end{cases} \tag{1.4}$$

for $t, s \in [t_1, t_3]$, *where*

$$\begin{aligned} u_1(t,s) &:= \frac{t-t_1}{2}(2s-t-t_1) + \frac{\gamma(t-t_1)}{2k}(s-t_1)^2(2t_2-t-t_1), \\ v_1(t,s) &:= u_1(t,s) + \frac{1}{2}(t-s)^2 = \frac{(s-t_1)^2}{2k}[k+\gamma(t-t_1)(2t_2-t-t_1)], \\ u_2(t,s) &:= \frac{t-t_1}{2k}(2t_2-t-t_1)[2\delta+\gamma(t_3-s)^2], \\ v_2(t,s) &:= u_2(t,s) + \frac{1}{2}(t-s)^2. \end{aligned}$$

Proof. First we check that $G(t,s)$ is well defined for all $(t,s) \in [t_1,t_3]\times[t_1,t_3]$. Clearly $u_1(s,s) = v_1(s,s)$ and $u_2(s,s) = v_2(s,s)$. At $s = t_2$,

$$\begin{aligned} u_1(t,t_2) &= \frac{t-t_1}{2}(2t_2-t-t_1) + \frac{\gamma(t-t_1)}{2k}(t_2-t_1)^2(2t_2-t-t_1) \\ &= \frac{t-t_1}{2k}(2t_2-t-t_1)[k+\gamma(t_2-t_1)^2] \\ &= \frac{t-t_1}{2k}(2t_2-t-t_1)[2\delta+\gamma(t_3-t_2)^2] \\ &= u_2(t,t_2). \end{aligned}$$

Likewise $v_1(t,t_2) = v_2(t,t_2)$.

Next, check that G satisfies the boundary conditions (1.2). For convenience we note that

$$\frac{\partial}{\partial t}G(t,s) = \begin{cases} s \in [t_1,t_2] & : \begin{cases} \frac{\gamma}{k}(s-t_1)^2(t_2-t)+s-t & : t \le s \\ \frac{\gamma}{k}(s-t_1)^2(t_2-t) & : s \le t \end{cases} \\ \\ s \in [t_2,t_3] & : \begin{cases} \frac{t_2-t}{k}\left[2\delta+\gamma(t_3-s)^2\right] & : t \le s \\ \frac{t_2-t}{k}\left[2\delta+\gamma(t_3-s)^2\right]-s+t & : s \le t \end{cases} \end{cases}$$

and

$$\frac{\partial^2}{\partial t^2}G(t,s) = \begin{cases} s \in [t_1,t_2] & : \begin{cases} -\frac{\gamma}{k}(s-t_1)^2-1 & : t < s \\ -\frac{\gamma}{k}(s-t_1)^2 & : t > s \end{cases} \\ s \in [t_2,t_3] & : \begin{cases} -\frac{1}{k}[2\delta+\gamma(t_3-s)^2] & : t < s \\ -\frac{1}{k}[2\delta+\gamma(t_3-s)^2]+1 & : t > s, \end{cases} \end{cases}$$

for fixed s; in the rest of this proof we will employ the shorthand G' and G'' for these two expressions. If $t = t_1$, then $t \le s$ for any $s \in [t_1,t_3]$, so that $G(t_1,s) = 0$. For $t = t_2$ and $s \in [t_1,t_2]$:

$$G'(t_2,s) = \frac{\partial}{\partial t}v_1(t_2,s) = \frac{\gamma}{k}(s-t_1)^2(t_2-t_2) = 0.$$

For $t = t_2$ and $s \in [t_2,t_3]$:

$$G'(t_2,s) = \frac{\partial}{\partial t}u_2(t_2,s) = \frac{t_2-t_2}{k}\left[2\delta+\gamma(t_3-s)^2\right] = 0.$$

Finally, consider the boundary condition at t_3. For $s \in [t_1, t_2]$:

$$\begin{aligned}
\gamma G(t_3,s) + \delta G''(t_3,s) &= \gamma v_1(t_3,s) + \delta \frac{\partial^2}{\partial t^2} v_1(t_3,s) \\
&= \frac{\gamma}{2k}(s-t_1)^2[k + \gamma(t_3-t_1)(2t_2-t_3-t_1)] \\
&\quad - \frac{\delta\gamma}{k}(s-t_1)^2 \\
&= \frac{\gamma(s-t_1)^2}{k}\left[\frac{k}{2} + \frac{\gamma}{2}(t_3-t_1)(2t_2-t_3-t_1) - \delta\right] \\
&= \frac{\gamma(s-t_1)^2}{k}\left[\frac{k}{2} - \frac{k}{2} + \delta - \delta\right] = 0.
\end{aligned}$$

For $s \in [t_2, t_3]$:

$$\begin{aligned}
\gamma G(t_3,s) + \delta G''(t_3,s) &= \gamma v_2(t_3,s) + \delta \frac{\partial^2}{\partial t^2} v_2(t_3,s) \\
&= \frac{\gamma(t_3-t_1)}{2k}(2t_2-t_3-t_1)[2\delta + \gamma(t_3-s)^2] \\
&\quad + \frac{\gamma k}{2k}(t_3-s)^2 + \frac{2\delta k}{2k} - \frac{2\delta}{2k}[2\delta + \gamma(t_3-s)^2] \\
&= \frac{1}{2k}\big\{(t_3-s)^2[\gamma k - 2\gamma\delta + 2\gamma\delta - \gamma k] + 2\delta k \\
&\quad - 4\delta^2 + 2\delta(2\delta - k)\big\} = 0.
\end{aligned}$$

Thus, G as in (1.4) satisfies the boundary conditions (1.2).

Now, for any function f continuous on $[t_1, t_3]$, define

$$x(t) := \int_{t_1}^{t_3} G(t,s)f(s)ds.$$

As shown above, this x satisfies the boundary conditions (1.2) via G. We will show that $x'''(t) = f(t)$ for $t \in [t_1, t_3]$. Note that for $t \in [t_1, t_2]$,

$$\begin{aligned}
x''(t) &= \left(\int_{t_1}^{t_2} + \int_{t_2}^{t_3}\right) \frac{\partial^2}{\partial t^2} G(t,s)f(s)ds \\
&= \left(\int_{t_1}^{t} + \int_{t}^{t_2}\right) \frac{\partial^2}{\partial t^2} G(t,s)f(s)ds - \frac{1}{k}\int_{t_2}^{t_3}[2\delta + \gamma(t_3-s)^2]f(s)ds \\
&= -\frac{\gamma}{k}\int_{t_1}^{t}(s-t_1)^2 f(s)ds + \int_{t}^{t_2}\left(-\frac{\gamma}{k}(s-t)^2 - 1\right)f(s)ds \\
&\quad - \frac{1}{k}\int_{t_2}^{t_3}[2\delta + \gamma(t_3-s)^2]f(s)ds \\
&= -\frac{\gamma}{k}\int_{t_1}^{t_2}(s-t_1)^2 f(s)ds - \frac{1}{k}\int_{t_2}^{t_3}[2\delta + \gamma(t_3-s)^2]f(s)ds \\
&\quad + \int_{t_2}^{t} f(s)ds,
\end{aligned}$$

so that $x'''(t) = 0 + 0 + f(t)$ using the Fundamental Theorem of Calculus. Likewise for $t \in [t_2, t_3]$,

$$x''(t) = -\frac{\gamma}{k}\int_{t_1}^{t_2}(s-t_1)^2 f(s)ds + \int_{t_2}^{t} f(s)ds - \frac{1}{k}\int_{t_2}^{t_3}[2\delta + \gamma(t_3-s)^2]f(s)ds$$

again implies that $x'''(t) = f(t)$. Therefore G as given in (1.4) is the Green's function for (1.1), (1.2). □

2. Positivity of Green's function

Theorem 2.1. *Assume $k > 0$. If*

$$t_2 - t_1 > t_3 - t_2, \tag{2.1}$$

then the Green's function as given in (1.4) *satisfies*

$$G(t_2, s) \geq G(t, s) > 0$$

on $(t_1, t_3] \times (t_1, t_3]$.

Proof. Note that $G(t, t_1) = 0$ for all $t \in [t_1, t_3]$. Since $k > 0$ and (2.1) holds, we must have $\delta > 0$ if $\gamma \geq 0$. We now proceed by cases on the two branches of the Green's function (1.4).

Case I: Let $s \in (t_1, t_2]$. Then $G(t_1, s) = u_1(t_1, s) = 0$, and

$$\frac{\partial}{\partial t}G(t,s) = \frac{\partial}{\partial t}u_1(t,s) = s - t + \frac{\gamma}{k}(s-t_1)^2(t_2 - t) \geq 0$$

for $t \in [t_1, \tau(s)]$; here

$$s \leq \tau(s) := \frac{ks + \gamma t_2(s-t_1)^2}{k + \gamma(s-t_1)^2} \leq t_2.$$

For $t \geq s$,

$$\frac{\partial}{\partial t}G(t,s) = \frac{\partial}{\partial t}v_1(t,s) = \frac{\gamma}{k}(s-t_1)^2(t_2 - t).$$

Therefore, G is increasing in t on (t_1, t_2) and decreasing in t on (t_2, t_3). It follows that $G(t_2, s) \geq G(t, s)$ for all $(t, s) \in [t_1, t_3] \times [t_1, t_2]$, and $G(t, s) > 0$ on $(t_1, t_3] \times (t_1, t_2]$ if $G(t_3, s) > 0$ for these s:

$$\begin{aligned} G(t_3, s) &= v_1(t_3, s) \\ &= \frac{(s-t_1)^2}{2k}[k + \gamma(t_3 - t_1)(2t_2 - t_3 - t_1)] \\ &= \frac{2\delta(s-t_1)^2}{2k}. \end{aligned}$$

As mentioned earlier, $\delta > 0$; consequently, $G(t_2, s) \geq G(t, s) > 0$ for $(t, s) \in (t_1, t_3] \times (t_1, t_2]$.

Case II: Now let $s \in [t_2, t_3]$. For $t \leq s$,

$$\frac{\partial}{\partial t}G(t,s) = \frac{\partial}{\partial t}u_2(t,s) = \frac{t_2 - t}{k}[2\delta + \gamma(t_3 - s)^2] > 0$$

if $t < t_2$. As a result we have that

$$0 = u_2(t_1, s) \le G(t, s)$$

for all $(t, s) \in [t_1, t_2] \times [t_2, t_3]$. For $t \in [t_2, s]$, G is then decreasing in t. If $t > s \ge t_2$, then $G(t, s) = v_2(t, s)$. Using

$$2t_2 - t_3 - t_1 > 0,$$

we see that $\frac{\partial^2}{\partial t^2} v_2(t, s) < 0$, so that

$$\frac{\partial}{\partial t} v_2(t, s) \le \frac{\partial}{\partial t} v_2(t, s)|_{t=s} = \frac{t_2 - s}{k}[2\delta + \gamma(t_3 - s)^2] \le 0$$

for $t > s \ge t_2$. Therefore G is increasing in t on $[t_1, t_2]$ and decreasing in t on $[t_2, t_3]$, with a maximum at $G(t_2, s)$. Again we check to see that $G(t_3, s) > 0$ for $s \in [t_2, t_3]$:

$$\begin{aligned} G(t_3, s) &= v_2(t_3, s) \\ &= \frac{t_3 - t_1}{2k}(2t_2 - t_3 - t_1)[2\delta + \gamma(t_3 - s)^2] + \frac{1}{2}(t_3 - s)^2. \end{aligned}$$

As a function of s we have

$$\frac{\partial}{\partial s} G(t_3, s) = \frac{\partial}{\partial s} v_2(t_3, s) = \frac{s - t_3}{k}[k + \gamma(t_3 - t_1)(2t_2 - t_3 - t_1)] \le 0$$

for $s \in [t_2, t_3]$; in other words, $G(t_3, t_3) \le G(t_3, s)$ for these s. But

$$G(t_3, t_3) = \frac{\delta}{k}(t_3 - t_1)(2t_2 - t_3 - t_1)$$

is positive, since by assumption $\delta > 0$ and

$$2t_2 - t_3 - t_1 > 0.$$

Therefore, $G(t_2, s) \ge G(t, s) > 0$ for $(t, s) \in (t_1, t_3] \times [t_2, t_3]$. Together with the first case, we have proved the theorem. □

Lemma 2.1. *Assume* (2.1) *holds and* $k > 0$. *For all* $t, s \in [t_1, t_3]$,

$$\ell(t)G(t_2, s) \le G(t, s) \le G(t_2, s) \tag{2.2}$$

where

$$\ell(t) := \frac{(t - t_1)(2t_2 - t - t_1)}{(t_2 - t_1)^2}. \tag{2.3}$$

Proof. As shown in Theorem 2.1, $G(t, s) \le G(t_2, s)$ for all $t, s \in [t_1, t_3]$. For the lower bound, we proceed by cases on the branches of the Green's function (1.4).

(*i*) $t \le s$, $s \in (t_1, t_2]$: Here $G(t, s)/G(t_2, s) = u_1(t, s)/v_1(t_2, s)$. For fixed t we have

$$\frac{\partial}{\partial s}\left(\frac{u_1(t, s)}{v_1(t_2, s)}\right) = \frac{2k(t - t_1)(s - t)}{-(s - t_1)^3[k + \gamma(t_2 - t_1)^2]} \le 0.$$

Thus $u_1(t,s)/v_1(t_2,s)$ is decreasing in s for $s \in (t_1,t_2]$, which implies

$$\frac{u_1(t,s)}{v_1(t_2,s)} \geq \frac{u_1(t,t_2)}{v_1(t_2,t_2)} = \frac{(t-t_1)(2t_2-t-t_1)}{(t_2-t_1)^2}.$$

(*ii*) $t \geq s$, $s \in (t_1,t_2]$: Notice that $G(t,s)/G(t_2,s) = v_1(t,s)/v_1(t_2,s)$, and

$$\frac{v_1(t,s)}{v_1(t_2,s)} - \frac{(t-t_1)(2t_2-t-t_1)}{(t_2-t_1)^2} = \frac{k(t_2-t)^2}{(t_2-t_1)^2[k+\gamma(t_2-t_1)^2]} \geq 0.$$

(*iii*) $t \leq s$, $s \in [t_2,t_3]$: In this case

$$\frac{G(t,s)}{G(t_2,s)} = \frac{u_2(t,s)}{u_2(t_2,s)} = \frac{(t-t_1)(2t_2-t-t_1)}{(t_2-t_1)^2}.$$

(*iv*) $t \geq s$, $s \in [t_2,t_3]$: Here we have $G(t,s)/G(t_2,s) = v_2(t,s)/u_2(t_2,s)$. Algebra leads to the fact that

$$\frac{v_2(t,s)}{u_2(t_2,s)} - \frac{(t-t_1)(2t_2-t-t_1)}{(t_2-t_1)^2} = \frac{k(t-s)^2}{(t_2-t_1)^2[2\delta+\gamma(t_3-s)^2]} \geq 0.$$

□

3. Existence of a solution

We are concerned with proving the existence of positive solutions of the third-order three-point generalized right focal boundary value problem

$$x'''(t) = f(t,x(t)) \text{ for all } t \in [t_1,t_3] \tag{3.1}$$

with boundary conditions

$$x(t_1) = x'(t_2) = 0 \tag{3.2}$$

$$\gamma x(t_3) + \delta x''(t_3) = 0$$

as in (1.2), where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, f nonnegative for $x \geq 0$, and $f(t,\cdot)$ does not vanish identically on any subset of $[t_1,t_3]$ of positive measure. The solutions of (3.1), (3.2) are the fixed points of the operator $\mathcal{A}$ defined by

$$\mathcal{A}x(t) = \int_{t_1}^{t_3} G(t,s)f(s,x(s))\,ds,$$

where $G(t,s)$ is the Green's function (1.4) for the homogeneous problem $x'''(t) = 0$ satisfying the same boundary conditions (3.2). Many of the proofs use techniques as in [16].

We will employ the following fixed point theorem due to Krasnoselskii [17]; first, a few definitions. A nonempty closed convex set $\mathcal{P}$ contained in a real Banach space E is called a cone if it satisfies the following two conditions:

(*i*) if $x \in \mathcal{P}$ and $\lambda \geq 0$ then $\lambda x \in \mathcal{P}$;
(*ii*) if $x \in \mathcal{P}$ and $-x \in \mathcal{P}$ then $x = 0$.

The cone $\mathcal{P}$ induces an ordering $\leq$ on E by $x \leq y$ if and only if $y - x \in \mathcal{P}$. An operator A is said to be completely continuous if it is continuous and compact (maps bounded sets into relatively compact sets).

Theorem 3.1. *Let E be a Banach space, $P \subseteq E$ be a cone, and suppose that Ω_1, Ω_2 are bounded open balls of E centered at the origin, with $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $\mathcal{A} : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|\mathcal{A}u\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$, or
(ii) $\|\mathcal{A}u\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$ and $\|\mathcal{A}u\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$

holds. Then $\mathcal{A}$ has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $\mathcal{B}$ denote the Banach space $C[t_1, t_3]$ with the norm

$$\|x\| = \sup_{t \in [t_1, t_3]} |x(t)|.$$

Let $h \in (0, t_3 - t_2)$. Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{x \in \mathcal{B} : x(t) \geq \ell(t_2 + h)\|x\|, t \in [t_2 - h, t_2 + h]\},$$

where ℓ is given in (2.3).

In the following discussion we will need the constants

$$\begin{aligned} m^{-1} &:= \int_{t_1}^{t_3} G(t_2, s)ds \qquad (3.3) \\ &= \frac{1}{6k}(t_2 - t_1)^2 \left\{(t_2 - t_1)[k + \gamma(t_2 - t_1)^2] + (t_3 - t_2)[6\delta + \gamma(t_3 - t_2)^2]\right\} \end{aligned}$$

and

$$\begin{aligned} r^{-1} &:= \ell(t_2 + h) \int_{t_2 - h}^{t_2 + h} G(t_2, s)\, ds \qquad (3.4) \\ &= \frac{(t_2 - t_1 + h)(t_2 - t_1 - h)}{(t_2 - t_1)^2} \left\{ \frac{k + \gamma(t_2 - t_1)^2}{6k} \left[(t_2 - t_1)^3 - (t_2 - t_1 - h)^3\right] \right. \\ &\qquad \left. + \frac{(t_2 - t_1)^2}{6k} \left[6h\delta + \gamma(t_2 - t_3 + h)^3 - \gamma(t_2 - t_3)^3\right] \right\}. \end{aligned}$$

Then the growth restrictions on f that will yield the existence of positive and multiple solutions are as follows:

(C_1) There exists a $p > 0$ such that $f(t, x) \leq mp$ for $t \in [t_1, t_3]$ and $0 \leq x \leq p$.
(C_2) There exists a $q > 0$ such that $f(t, x) \geq rx$ for $t \in [t_2 - h, t_2 + h]$ and $q\ell(t_2 + h) \leq x \leq q$, for $h \in (0, t_3 - t_2)$.

Theorem 3.2. *Assume* (2.1) *holds, and suppose there exist positive numbers $p \neq q$ such that condition (C_1) is satisfied with respect to p and condition (C_2) is satisfied with respect to q. Then* (3.1), (3.2) *has a positive solution x such that $||x||$ lies between p and q.*

Proof. We first establish that $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$. For $x \in \mathcal{P}$, clearly $\mathcal{A}x \in \mathcal{B}$. Let $h \in (0, t_3 - t_2)$ and $t \in [t_2 - h, t_2 + h]$; then using Lemma 2.1 we see that

$$\begin{aligned}\mathcal{A}x(t) &= \int_{t_1}^{t_3} G(t,s) f(s, x(s))\, ds \geq \int_{t_1}^{t_3} \ell(t) G(t_2, s) f(s, x(s))\, ds \\ &= \ell(t_2 + h) ||\mathcal{A}x||,\end{aligned}$$

since

$$\min_{t \in [t_2 - h, t_2 + h]} \ell(t) = \min\{\ell(t_2 - h), \ell(t_2 + h)\}$$

and $\ell(t_2 + h) = \ell(t_2 - h)$ for all $h \in (0, t_3 - t_2)$, ℓ as in (2.3). It follows that $Ax \in \mathcal{P}$ whenever $x \in \mathcal{P}$; i.e., $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$.

Now without loss of generality, we may assume $0 < p < q$. Define bounded open balls

$$\Omega_p = \{x \in \mathcal{B} : ||x|| < p\},$$

and

$$\Omega_q = \{x \in \mathcal{B} : ||x|| < q\}.$$

Then $0 \in \Omega_p \subset \Omega_q$. For $x \in \mathcal{P} \cap \partial\Omega_p$ so that $||x|| = p$, we have

$$\begin{aligned}||\mathcal{A}x|| &= \int_{t_1}^{t_3} G(t_2, s) f(s, x(s))\, ds \\ &\leq mp \int_{t_1}^{t_3} G(t_2, s)\, ds = p = ||x||\end{aligned}$$

using (C_1) and (3.3). Thus, $||\mathcal{A}x|| \leq ||x||$ for $x \in \mathcal{P} \cap \partial\Omega_p$.

Similarly, let $x \in \mathcal{P} \cap \partial\Omega_q$, so that $||x|| = q$. Then

$$\min_{t \in [t_2 - h, t_2 + h]} x(t) \geq ||x|| \ell(t_2 + h).$$

As a result, $q\ell(t_2 + h) \leq x(s) \leq q$ for $s \in [t_2 - h, t_2 + h]$, and we have for $t \in [t_2 - h, t_2 + h]$

$$\begin{aligned}||\mathcal{A}x|| &= \int_{t_1}^{t_3} G(t_2, s) f(s, x(s))\, ds \geq \int_{t_2 - h}^{t_2 + h} G(t_2, s) f(s, x(s))\, ds \\ &\geq r \int_{t_2 - h}^{t_2 + h} G(t_2, s) x(s)\, ds \geq q = ||x||\end{aligned}$$

by (C_2) and (3.4). Consequently, $||\mathcal{A}x|| \geq ||x||$ for $x \in \mathcal{P} \cap \partial\Omega_q$. By Theorem 3.1, $\mathcal{A}$ has a fixed point $x \in \mathcal{P} \cap (\overline{\Omega}_q \setminus \Omega_p)$, which is a positive solution of (3.1), (3.2) such that $p \leq ||x|| \leq q$. □

Example 3.1. We illustrate Theorem 3.2 with specific parameter values. Let $t_1 = 0$, $t_2 = \frac{11}{20}$, $t_3 = 1$, $\gamma = 1$, and $\delta = \frac{1}{2}$; then $k = \frac{9}{10}$ and $m = \frac{864000}{101761}$.

By (1.4), the Green's function G is given by

$$G(t,s) = \begin{cases} s \in [t_1, t_2] & : \begin{cases} \frac{t}{2}(2s-t) + \frac{5}{9}ts^2(\frac{11}{10}-t) & : t \le s \\ \frac{5}{9}s^2\left(\frac{9}{10} + t(\frac{11}{10}-t)\right) & : t \ge s \end{cases} \\ \\ s \in [t_2, t_3] & : \begin{cases} \frac{5}{9}t(\frac{11}{10}-t)(1+(1-s)^2) & : t \le s \\ \frac{1}{2}(t-s)^2 + \frac{5}{9}t(\frac{11}{10}-t)(1+(1-s)^2) & : t \ge s. \end{cases} \end{cases}$$

If $h = \frac{1}{20}$, then $r = \frac{4356000}{82667}$ and $\ell(t_2+h) = \frac{120}{121}$. Let $p = \frac{10}{m}$ and $q = \frac{3}{2}$; for $t \in [0,1]$, let

$$f(t,x) := \begin{cases} 8x\sin(\pi t) & : x \in [0,p] \\ L(x)\sin(\pi t) & : x \in [p, q\ell(t_2+h)] \\ 60x\sin(\pi t) & : x \in [q\ell(t_2+h), \infty) \end{cases}$$

where $L(x) := 8p + (x-p)\frac{60q\ell(t_2+h)-8p}{q\ell(t_2+h)-p}$. Note that f is continous and nonnegative valued for $x \ge 0$.

(C_1) For $t \in [t_1, t_3]$ and $0 \le x \le p$, $f(t,x) = 8x\sin(\pi t) \le 8 < 10 = mp$.

(C_2) For $t \in [t_2 - h, t_2 + h]$ and $q\ell(t_2+h) \le x \le q$, $f(t,x) = 60x\sin(\pi t) \ge 60x\sin(\frac{3\pi}{5}) \ge 57x > rx$.

We conclude from Theorem 3.2 that for these parameter values, (3.1), (3.2) has a positive solution x such that $\frac{101761}{86400} \le ||x|| \le \frac{3}{2}$.

4. Existence of three solutions

In this section we prove the existence of at least three solutions to (3.1), (3.2) using another fixed point theorem, this one due to Leggett and Williams, whose proof can be found in Guo and Lakshmikantham [15], or Leggett and Williams [18].

A map ψ is a nonnegative continuous concave functional on the cone $\mathcal{P}$ if it satisfies the following conditions:

(i) $\psi : \mathcal{P} \to [0,\infty)$ is continuous;

(ii) $\psi(tx + (1-t)y) \ge t\psi(x) + (1-t)\psi(y)$ for all $x, y \in \mathcal{P}$ and $0 \le t \le 1$.

Let

$$\mathcal{P}_c := \{x \in \mathcal{P} : \|x\| < c\}$$

and

$$\mathcal{P}(\psi, a, b) := \{x \in \mathcal{P} : a \le \psi(x),\ \|x\| \le b\}.$$

Theorem 4.1. *Let $\mathcal{P}$ be a cone in the real Banach space E, $A : \overline{\mathcal{P}_c} \to \overline{\mathcal{P}_c}$ be completely continuous and ψ be a nonnegative continuous concave functional on $\mathcal{P}$ with $\psi(x) \le \|x\|$ for all $x \in \overline{\mathcal{P}_c}$. Suppose there exists $0 < a < b < d \le c$ such that the following conditions hold:*

(i) $\{x \in \mathcal{P}(\psi, b, d) : \psi(x) > b\} \ne \emptyset$ *and* $\psi(Ax) > b$ *for all* $x \in \mathcal{P}(\psi, b, d)$;

(ii) $\|Ax\| < a$ *for* $\|x\| \le a$;

(iii) $\psi(Ax) > b$ *for* $x \in \mathcal{P}(\psi, b, c)$ *with* $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2, and x_3 in $\overline{\mathcal{P}_c}$ satisfying:

$$\|x_1\| < a, \quad \psi(x_2) > b, \quad a < \|x_3\| \text{ with } \psi(x_3) < b.$$

Let the Banach space $\mathcal{B} = C[t_1, t_3]$ be endowed with the sup norm, and define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \{x \in \mathcal{B} : x \text{ concave and nonnegative valued on } [t_1, t_3]\}.$$

Let the nonnegative continuous concave functional $\psi : \mathcal{P} \to [0, \infty)$ by defined by

$$\psi(x) = \min_{t \in [t_2 - h, t_2 + h]} x(t), \quad x \in \mathcal{P}. \tag{4.1}$$

Note that for $x \in \mathcal{P}$, $\psi(x) \le \|x\|$.

Remark 4.1. In the main existence theorem to follow, we will need the following constant for $h \in (0, t_3 - t_2)$:

$$N := \min \left\{ \int_{t_2 - h}^{t_2 + h} G(t_2 - h, s) ds, \ \int_{t_2 - h}^{t_2 + h} G(t_2 + h, s) ds \right\}.$$

It can be shown via very tedious calculations that

$$\int_{t_2 - h}^{t_2 + h} G(t_2 + h, s) ds - \int_{t_2 - h}^{t_2 + h} G(t_2 - h, s) ds = \frac{h^3}{3}.$$

As a result we have

$$N = \int_{t_2 - h}^{t_2 + h} G(t_2 - h, s) ds. \tag{4.2}$$

Also note that $\ell(t_2 + h) < 1$, as needed in the statement of the next theorem.

Theorem 4.2. *Assume* (2.1) *holds. Let $f : [t_1, t_3] \times [0, \infty) \to [0, \infty)$ be continuous, and $h \in (0, t_3 - t_2)$. Suppose that there exist positive constants a, b, c satisfying*

$$0 < a < b < b/\ell(t_2 + h) \le c$$

such that

- (D_1) *$f(t, x) < ma$ for $t \in [t_1, t_3]$, $x \in [0, a]$,*
- (D_2) *$f(t, x) \ge b/N$ for $t \in [t_2 - h, t_2 + h]$, $x \in [b, b/\ell(t_2 + h)]$,*
- (D_3) *$f(t, x) \le mc$ for $t \in [t_1, t_3]$, $x \in [0, c]$,*

where ℓ, m and N are as defined in (2.3), (3.3), *and* (4.2), *respectively. Then the boundary value problem* (3.1), (3.2) *has at least three positive solutions x_1, x_2, x_3 satisfying*

$$\|x_1\| < a, \qquad b < \psi(x_2), \qquad \|x_3\| > a \text{ with } \psi(x_3) < b.$$

Proof. Define the operator $A : \mathcal{P} \to \mathcal{B}$ by

$$Ax(t) = \int_{t_1}^{t_3} G(t, s) f(s, x(s)) ds.$$

Note that if $x \in \mathcal{P}$, the fact that f is nonnegative and Theorem 2.1 imply that $Ax(t) \ge 0$ for $t \in [t_1, t_3]$. Since $\frac{\partial^2}{\partial t^2} G(t, s) < 0$ for $(t, s) \in (t_1, t_3)^2$ so that $(Ax)''(t) \le 0$ for $t \in (t_1, t_3)$, we see that $Ax \in \mathcal{P}$; i.e., $A : \mathcal{P} \to \mathcal{P}$. Moreover, A is completely continuous.

We now show that all of the conditions of Theorem 4.1 are satisfied. For all $x \in \mathcal{P}$ we have $\psi(x) \le \|x\|$. If $x \in \overline{\mathcal{P}_c}$, then $\|x\| \le c$ and assumption (D_3) implies $f(t, x(t)) \le mc$ for $t \in [t_1, t_3]$. As a result,

$$\begin{aligned} \|Ax\| &= \sup_{t\in[t_1,t_3]} \int_{t_1}^{t_3} G(t,s)f(s,x(s))ds \le \sup_{t\in[t_1,t_3]} \left(\int_{t_1}^{t_3} G(t,s)ds\right) mc \\ &\le mc \int_{t_1}^{t_3} G(t_2,s)ds = c. \end{aligned}$$

Therefore $A : \overline{\mathcal{P}_c} \to \overline{\mathcal{P}_c}$. In the same way, if $x \in \mathcal{P}_a$, then assumption (D_1) yields $f(t, x(t)) < ma$ for $t \in [t_1, t_3]$; as in the argument above, it follows that $A : \overline{\mathcal{P}_a} \to \mathcal{P}_a$. Hence, condition (ii) of Theorem 4.1 is satisfied.

To check condition (i) of Theorem 4.1, choose $x_P(t) \equiv b/\ell(t_2 + h)$ for $t \in [t_1, t_3]$, where ℓ is given in (2.3). Then $x_P \in \mathcal{P}(\psi, b, b/\ell(t_2 + h))$ and $\psi(x_P) = \psi(b/\ell(t_2+h)) > b$, so that $\{x \in \mathcal{P}(\psi, b, b/\ell(t_2+h)) : \psi(x) > b\} \ne \emptyset$. Consequently, if $x \in \mathcal{P}(\psi, b, b/\ell(t_2 + h))$, then $b \le x(s) \le b/\ell(t_2 + h)$ for $s \in [t_2 - h, t_2 + h]$. From assumption (D_2) we have that

$$f(t, x(t)) \ge b/N$$

for $t \in [t_2 - h, t_2 + h]$; by the definitions of ψ and the cone $\mathcal{P}$, and the concavity of G, we must have one of two possibilities: $\psi(Ax(t)) = Ax(t_2-h)$ or $\psi(Ax(t)) = Ax(t_2 + h)$. Following the discussion in Remark 4.1, we see that

$$\begin{aligned} \psi(Ax) &= \min_{[t_2-h,t_2+h]} Ax(t) = Ax(t_2 \pm h) = \int_{t_1}^{t_3} G(t_2-h,s)f(s,x(s))ds \\ &> \int_{t_2-h}^{t_2+h} G(t_2-h,s)f(s,x(s))ds \\ &\ge \frac{b}{N}\int_{t_2-h}^{t_2+h} G(t_2-h,s)ds \ge \frac{b}{N}N = b, \end{aligned}$$

for N as in (4.2). Consequently we have

$$\psi(Ax) > b, \quad x \in \mathcal{P}(\psi, b, b/\ell(t_2 + h)),$$

so that condition (i) of Theorem 4.1 holds.

Lastly we consider Theorem 4.1 (iii). Suppose $x \in \mathcal{P}(\psi, b, c)$ with $\|Ax\| > b/\ell(t_2 + h)$. Then, using the definition of ψ and Lemma 2.2, we see that

$$\begin{aligned} \psi(Ax) &= \min_{t\in[t_2-h,t_2+h]} Ax(t) = \min_{t\in[t_2-h,t_2+h]} \int_{t_1}^{t_3} G(t,s)f(s,x(s))ds \\ &\ge \ell(t_2+h)\int_{t_1}^{t_3} G(t_2,s)f(s,x(s))ds \\ &\ge \ell(t_2+h)\|Ax\| > b\ell(t_2+h)/\ell(t_2+h) = b. \end{aligned}$$

□

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