DISCRETE THIRD-ORDER THREE-POINT RIGHT FOCAL BOUNDARY VALUE PROBLEMS

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Abstract. We are concerned with the discrete right-focal boundary value problem $\Delta^3 x(t) = f(t, x(t+1))$, $x(t_1) = \Delta x(t_2) = \Delta^2 x(t_3) = 0$, and the eigenvalue problem $\Delta^3 x(t) = \lambda a(t) f(x(t + 1))$ with the same boundary conditions, where $t_1 < t_2 < t_3$. Under various assumptions on $f$, $a$ and $\lambda$ we prove the existence of positive solutions of both problems by applying a fixed point theorem.

1. Introduction

In this paper, we are concerned with the existence of positive solutions to the discrete third-order three-point eigenvalue problem:

$$\Delta^3 x(t) = \lambda a(t) f(x(t + 1)) \quad \text{for all } t \in [t_1, t_3 - 1]$$

$$x(t_1) = \Delta x(t_2) = \Delta^2 x(t_3) = 0,$$

and the boundary value problem

$$\Delta^3 x(t) = f(t, x(t + 1)) \quad \text{for all } t \in [t_1, t_3 - 1]$$

$$x(t_1) = \Delta x(t_2) = \Delta^2 x(t_3) = 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(x) \geq 0$ for $x \geq 0$, with $[a, b] := \{a, a + 1, a + 2, \ldots, b\}$. A solution of (2), (3) is nonnegative on $[t_1, t_3 + 2]$, nondecreasing on $[t_1, t_2]$, and nonincreasing on $[t_2, t_3 + 2]$. In this discrete case, conditions on $f$ were imposed to yield at least three positive solutions applying the Leggett-Williams fixed point theorem [8] and its generalization, the so-called five functionals fixed point theorem [9]. On the unit interval, the continuous third-order three-point

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right-focal case was introduced in [7], while [13] covered the nth-order two-point right-focal eigenvalue problem.

The literature on positive solutions to boundary value problems is extensive; for a small sampling, see [1, 2, 3, 4, 11, 12, 19]. [5] gives a good overview for much of the recent work which has been done and the methods used.

2. Eigenvalue Intervals

Consider the nonlinear discrete right-focal eigenvalue problem
\[
\Delta^3 x(t) = \lambda a(t) f(x(t + 1)), \quad t_1 \leq t \leq t_3 - 1, \\
x(t_1) = \Delta x(t_2) = \Delta^2 x(t_3) = 0,
\]
where \( t_1 < t_2 < t_3 \) are distinct integers. The corresponding Green’s function for the homogeneous problem \( \Delta^3 x(t) = 0 \) satisfying the boundary conditions (5) is given [6] by
\[
G(t, s) = \begin{cases} 
\frac{1}{2}(t - t_1)(2s - t - t_1 + 3) : t \leq s + 1 \\
\frac{1}{2}(s - t_1 + 2)^2 : t \geq s + 1 
\end{cases}
\]
\[
G(t, s) = \begin{cases} 
\frac{1}{2}(t - t_1)(2t_2 - t - t_1 + 1) \\
\frac{1}{2}(t - t_1)(2t_2 - t - t_1 + 1) + \frac{1}{2}(t - s - 1)^2 
\end{cases}
\]
Recall the so-called discrete factorial notation \( t^{(j)} := t(t - 1) \cdots (t - j + 1) \).

Remark. By [6] if \( t_2 - t_1 - 1 > t_3 - t_2 \),
\[
G(t_2, s) \geq G(t, s) > 0
\]
for \( t \in (t_1, t_3 + 2] \), \( s \in [t_1, t_3 - 1] \). Thus throughout this paper we assume that
\[
t_2 - t_1 - 1 > t_3 - t_2.
\]

Furthermore we have the assumptions
(A1) \( a(t) \) is a nonnegative function defined on \( [t_1, t_3 - 1] \) satisfying
\[
0 < \sum_{s=t_1}^{t_3-1} G(t_2, s)a(s) < \infty,
\]
where, using (6),

\[
G(t_2, s) = \begin{cases} 
\frac{1}{2}(s - t_1 + 2) : s \in [t_1, t_2 - 1] \\
\frac{1}{2}(t_2 - t_1 + 1) : s \in [t_2 - 1, t_3 - 1]
\end{cases}
\]

(A2) \( f : [0, \infty) \to [0, \infty) \) is continuous such that both

\[
f_0 := \lim_{x \to 0^+} \frac{f(x)}{x} \quad \text{and} \quad f_\infty := \lim_{x \to \infty} \frac{f(x)}{x}
\]
exist.

**Lemma 1.** For all \( t \in [t_1, t_3 + 2] \) and all \( s \in [t_1, t_3 - 1] \),

\[
g(t)G(t_2, s) \leq G(t, s) \leq G(t_2, s)
\]

where

\[
g(t) := \min \left\{ \frac{t - t_1}{t_2 - t_1}, \frac{t_3 - t + 2}{t_2 - t_1} \right\}.
\]

**Proof.** As noted in the preceding remark we have from [6] that \( G(t, s) \leq G(t_2, s) \) for all \( t \in [t_1, t_3 + 2], s \in [t_1, t_3 - 1] \). For the lower bound, we proceed by cases on the branches of the Green’s function (6).

(i) \( t_1 \leq t \leq s + 1 \leq t_2 \): Here \( G(t, s) = \frac{1}{2}(t - t_1)(2s - t - t_1 + 3), \) \( G(t_2, s) = \frac{1}{2}(s - t_1 + 2)(2) \). For these \( t, s \) we have

\[
\frac{(s - t_1 + 2)(2)}{t_2 - t_1} \leq s - t_1 + 2 \leq 2s - t - t_1 + 3,
\]

which implies

\[
\left( \frac{t - t_1}{t_2 - t_1} \right) G(t_2, s) \leq G(t, s).
\]

(ii) \( t_1 \leq s \leq t - 1 \leq t_2 - 1 \): Since \( G(t, s) = \frac{1}{2}(s - t_1 + 2)(2) = G(t_2, s) \) and \( t \leq t_2 \), it follows that

\[
\left( \frac{t - t_1}{t_2 - t_1} \right) G(t_2, s) \leq G(t, s).
\]

(iii) \( t_1 \leq s \leq t_2 - 1 \leq t - 1 \leq t_3 + 1 \): As in case (ii), \( G(t, s) = \frac{1}{2}(s - t_1 + 2)(2) = G(t_2, s) \); as \( t \in [t_2, t_3 + 2] \), we have

\[
\left( \frac{t_3 - t + 2}{t_3 - t_2 + 2} \right) G(t_2, s) \leq G(t, s).
\]
(iv) \( t_1 \leq t \leq t_2 \leq s + 1 \leq t_3 \): In this case \( G(t, s) = \frac{1}{2}(t - t_1)(2t_2 - t - t_1 + 1) \) and \( G(t_2, s) = \frac{1}{2}(t_2 - t_1 + 1)^{(2)} \). Simplification of terms verifies
\[
\left( \frac{t - t_1}{t_2 - t_1} \right) G(t_2, s) \leq G(t, s).
\]

(v) \( t_2 \leq t \leq s + 1 \leq t_3 \): As is (iv), \( G(t, s) = \frac{1}{2}(t - t_1)(2t_2 - t - t_1 + 1) \) and \( G(t_2, s) = \frac{1}{2}(t_2 - t_1 + 1)^{(2)} \). Define
\[
w(t) := \frac{1}{2}(t - t_1)(2t_2 - t - t_1 + 1) - \frac{1}{2} \left( \frac{t_3 - t + 2}{t_3 - t_2 + 2} \right) (t_2 - t_1 + 1)^{(2)} \tag{11}
\]
\[
eq G(t, s) - \left( \frac{t_3 - t + 2}{t_3 - t_2 + 2} \right) G(t_2, s).
\]

Now \( w(t_2) = 0, \Delta w(t_2) = w(t_2 + 1) > 0 \), and \( w(t_3 + 2) = G(t_3 + 2, s) > 0 \) by (7). Since \( w \) is concave down, \( w(t) \geq 0 \) on \([t_2, t_3 + 2]\), hence
\[
\left( \frac{t_3 - t + 2}{t_3 - t_2 + 2} \right) G(t_2, s) \leq G(t, s).
\]

(vi) \( t_2 \leq s + 1 \leq t \leq t_3 + 2 \): Note that \( G(t_2, s) = \frac{1}{2}(t_2 - t_1 + 1)^{(2)} \), while \( G(t, s) = \frac{1}{2}(t - t_1)(2t_2 - t - t_1 + 1) + \frac{1}{2}(t - s - 1)^{(2)} \geq \frac{1}{2}(t - t_1)(2t_2 - t - t_1 + 1) \); consequently, the employment of \( w \) as in (11) yields
\[
\left( \frac{t_3 - t + 2}{t_3 - t_2 + 2} \right) G(t_2, s) \leq G(t, s).
\]

To establish eigenvalue intervals we will employ the following fixed point theorem due to Krasnoselskii [17].

**Theorem 1.** Let \( E \) be a Banach space, \( K \subseteq E \) be a cone, and suppose that \( \Omega_1, \Omega_2 \) are bounded open balls of \( E \) centered at the origin with \( \overline{\Omega}_1 \subset \Omega_2 \). Suppose further that \( A : K \cap (\Omega_2 \setminus \Omega_1) \to K \) is a completely continuous operator such that either

(i) \( \|Au\| \leq \|u\|, u \in K \cap \partial \Omega_1 \) and \( \|Au\| \geq \|u\|, u \in K \cap \partial \Omega_2 \), or

(ii) \( \|Au\| \geq \|u\|, u \in K \cap \partial \Omega_1 \) and \( \|Au\| \leq \|u\|, u \in K \cap \partial \Omega_2 \)

holds. Then \( A \) has a fixed point in \( K \cap (\Omega_2 \setminus \Omega_1) \).

Let \( B = \{x : [t_1, t_3 + 2] \to \mathbb{R}\} \) denote the Banach space with the norm \( \|x\| = \sup_{t \in [t_1, t_3 + 2]} |x(t)| \). Define the cone \( P \subset B \) by
\[
P = \{x \in B : x(t) \geq g(t)\|x\|, t \in [t_1, t_3 + 2]\},
\]
where $g$ is given in (10). Let $h \in (0, t_3 - t_2 - 1)$ be chosen such that

$$\sum_{s=t_2-h}^{t_2+h} G(t_2, s)a(s) > 0.$$  \hspace{1cm} (12)

For $h$ as in (12), set

$$u(h) := 1 - \frac{h + 1}{t_3 - t_2 + 2}.$$  \hspace{1cm} (13)

**Theorem 2.** Suppose (A1) and (A2) hold. Then for each $\lambda$ satisfying

$$\frac{1}{\sum_{s=t_2-h}^{t_2+h} G(t_2, s)a(s)} < \lambda < \frac{1}{\sum_{s=t_1}^{t_3-1} G(t_2, s)a(s)}$$  \hspace{1cm} (14)

there exists at least one solution of (4), (5) in $P$.

**Proof.** Let $\lambda$ be as in (14) and let $\epsilon > 0$ be such that

$$\frac{1}{\sum_{s=t_2-h}^{t_2+h} G(t_2, s)a(s)} < \lambda < \frac{1}{\sum_{s=t_1}^{t_3-1} G(t_2, s)a(s)}.$$  \hspace{1cm} (15)

Since $x$ is a solution of (4), (5) if and only if

$$x(t) = \lambda \sum_{s=t_1}^{t_3-1} G(t, s)a(s)f(x(s + 1)), \quad t \in [t_1, t_3 + 2],$$

we define the operator $T : P \to B$ by

$$(Tx)(t) := \lambda \sum_{s=t_1}^{t_3-1} G(t, s)a(s)f(x(s + 1)), \quad x \in P.$$  \hspace{1cm} (16)

We seek a fixed point of $T$ in $P$ by establishing the hypotheses of Theorem 1. First, if $x \in P$ then by (9) we have

$$\|Tx\| = \lambda \sum_{s=t_1}^{t_3-1} G(t_2, s)a(s)f(x(s + 1))$$

and

$$(Tx)(t) = \lambda \sum_{s=t_1}^{t_3-1} G(t, s)a(s)f(x(s + 1))$$

$$\geq \lambda g(t) \sum_{s=t_1}^{t_3-1} G(t_2, s)a(s)f(x(s + 1)) = g(t)\|Tx\|.$$  \hspace{1cm} (17)

Therefore $T : P \to P$. Moreover, $T$ is completely continuous by a typical application of the Ascoli-Arzela Theorem.
Now consider $f_0$. There exists an $H_1 > 0$ such that $f(x) \leq (f_0 + \epsilon)x$ for $0 < x \leq H_1$ by the definition of $f_0$. Pick $x \in \mathcal{P}$ with $\|x\| = H_1$. Using (9) we have

\[
(Tx)(t) = \lambda \sum_{s=t_1}^{t_2-1} G(t, s) \alpha(s) f(x(s+1)) \\
\leq \lambda (f_0 + \epsilon) \|x\| \sum_{s=t_1}^{t_2-1} G(t_2, s) \alpha(s) \\
\leq \|x\|
\]

from the right side of (15). As a result, $\|Tx\| \leq \|x\|$. Thus, take

\[
\Omega_1 := \{x \in \mathcal{B} : \|x\| < H_1\}
\]

so that $\|Tx\| \leq \|x\|$ for $x \in \mathcal{P} \cap \partial \Omega_1$.

Next consider $f_\infty$. Again by definition there exists an $H_2 > H_1$ such that $f(x) \geq (f_\infty - \epsilon)x$ for $x \geq H_2$. If $x \in \mathcal{P}$ with $\|x\| = H_2$, then for $s \in [t_2 - h, t_2 + h]$, where $h$ is as in (12) and $u(h)$ as in (13), we have

\[
x(s+1) \geq g(s+1) \|x\| = g(s+1) H_2 \geq g(t_2 + h + 1) H_2 \geq u(h) H_2.
\]

(17)

Define $\Omega_2 := \{x \in \mathcal{B} : \|x\| < H_2\}$; using (17), we get

\[
\|Tx\| = \lambda \sum_{s=t_1}^{t_2-1} G(t_2, s) \alpha(s) f(x(s+1)) \\
\geq \lambda \sum_{s=t_2-h}^{t_2+h} G(t_2, s) \alpha(s) f(x(s+1)) \\
\geq \lambda (f_\infty - \epsilon) \sum_{s=t_2-h}^{t_2+h} G(t_2, s) \alpha(s) x(s+1) \\
\geq \lambda (f_\infty - \epsilon) u(h) H_2 \sum_{s=t_2-h}^{t_2+h} G(t_2, s) \alpha(s) \\
\geq H_2 \\
= \|x\|,
\]

where the penultimate line follows from the left side of (15). Hence we have shown that

\[
\|Tx\| \geq \|x\|, \quad x \in \mathcal{P} \cap \partial \Omega_2.
\]

An application of Theorem 1 validates the conclusion of the theorem. \qed
Theorem 3. Suppose (A1) and (A2) hold. Then for each $\lambda$ satisfying
\[
\frac{1}{f_0(h) \sum_{s=t_2-h}^{t_2+h} G(t_2, s)a(s)} < \lambda < \frac{1}{f_\infty \sum_{s=t_1}^{t_3-1} G(t_2, s)a(s)}
\]
(18)
there exists at least one solution of (4), (5) in $P$.

Proof. Let $\lambda$ be as in (18) and let $\eta > 0$ be such that
\[
\frac{1}{(f_0 - \eta) u(h) \sum_{s=t_2-h}^{t_2+h} G(t_2, s)a(s)} \leq \lambda \leq \frac{1}{(f_\infty + \eta) \sum_{s=t_1}^{t_3-1} G(t_2, s)a(s)}.
\]
(19)
Let $T$ be the completely continuous, cone-preserving operator defined in (16). We seek a fixed point of $T$ in $P$ by establishing the hypotheses of Theorem 1.

First, consider $f_0$. There exists an $H_1 > 0$ such that $f(x) \leq (f_0 - \eta)x$ for $0 < x \leq H_1$ by the definition of $f_0$. Pick $x \in P$ with $\|x\| = H_1$. For $s \in [t_2 - h, t_2 + h]$, where $h$ is as in (12) and $u(h)$ as in (13), we have
\[
x(s + 1) \geq g(s + 1) \|x\| = g(s + 1)H_1 \geq g(t_2 + h + 1)H_1 = u(h)H_1.
\]
(20)
Using the left side of (19) and (20), we get
\[
\|Tx\| = \lambda \sum_{s=t_1}^{t_3-1} G(t_2, s)a(s)f(x(s+1))
\]
\[
\geq \lambda \sum_{s=t_2-h}^{t_2+h} G(t_2, s)a(s)f(x(s+1))
\]
\[
\geq \lambda(f_0 - \eta) \sum_{s=t_2-h}^{t_2+h} G(t_2, s)a(s)x(s+1)
\]
\[
\geq \lambda(f_0 - \eta)u(h)H_1 \sum_{s=t_2-h}^{t_2+h} G(t_2, s)a(s)
\]
\[
\geq H_1
\]
\[
= \|x\|.
\]
Therefore $\|Tx\| \geq \|x\|$. This prompts us to define
\[
\Omega_1 := \{x \in B : \|x\| < H_1\},
\]
whereby our work above confirms
\[
\|Tx\| \geq \|x\|, \quad x \in P \cap \partial \Omega_1.
\]
Next consider \( f_\infty \). Again by definition there exists an \( \overline{H}_2 > H_1 \) such that \( f(x) \leq (f_\infty + \eta)x \) for \( x \geq \overline{H}_2 \). If \( f \) is bounded, there exists \( M > 0 \) with \( f(x) \leq M \) for all \( x \in (0, \infty) \). Let
\[
H_2 := \max\{2\overline{H}_2, \lambda M \sum_{s=t_1}^{t_2-1} G(t_2, s)a(s)\}.
\]
If \( x \in \mathcal{P} \) with \( \|x\| = H_2 \), then we have
\[
\|Tx\| = \lambda \sum_{s=t_1}^{t_2-1} G(t_2, s)a(s)f(x(s + 1)) \\
\leq \lambda M \sum_{s=t_1}^{t_2-1} G(t_2, s)a(s) \\
\leq H_2 \\
= \|x\|.
\]
As a result, \( \|Tx\| \leq \|x\| \). Thus, take
\[
\Omega_2 := \{x \in \mathcal{B} : \|x\| < H_2\}
\]
so that \( \|Tx\| \leq \|x\| \) for \( x \in \mathcal{P} \cap \partial \Omega_2 \). If \( f \) is unbounded, take \( H_2 := \max\{2H_1, \overline{H}_2\} \) such that \( f(x) \leq f(H_2) \) for \( 0 < x \leq H_2 \). If \( x \in \mathcal{P} \) with \( \|x\| = H_2 \), then we have
\[
\|Tx\| = \lambda \sum_{s=t_1}^{t_2-1} G(t_2, s)a(s)f(x(s + 1)) \\
\leq \lambda \sum_{s=t_1}^{t_2-1} G(t_2, s)a(s)f(H_2) \\
\leq \lambda (f_\infty + \eta)H_2 \sum_{s=t_1}^{t_2-1} G(t_2, s)a(s) \\
\leq H_2 \\
= \|x\|.
\]
where the penultimate line follows from the left side of (19). Hence we have shown that
\[
\|Tx\| \leq \|x\|, \quad x \in \mathcal{P} \cap \partial \Omega_2
\]
if we take
\[
\Omega_2 := \{x \in \mathcal{B} : \|x\| < H_2\}.
\]
Once again an application of Theorem 1 validates the conclusion of the theorem. \( \square \)
3. Existence of One Positive Solution

In the remaining sections we are concerned with proving the existence of positive solutions of the discrete third-order three-point right-focal boundary value problem

$$\Delta^3 x(t) = f(t, x(t+1)) \text{ for all } t \in [t_1, t_3 - 1]$$

with boundary conditions

$$x(t_1) = \Delta x(t_2) = \Delta^2 x(t_3) = 0$$

as in (3), where $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $f$ nonnegative for $x \geq 0$. Defined on $[t_1, t_3 + 2]$, the solutions of (2), (3) are the fixed points of the operator $A$ defined by

$$Ax(t) = \sum_{s=t_1}^{t_3-1} G(t, s)f(s, x(s + 1)),$$

where $G(t, s)$ is the Green's function (6) for the for the homogeneous problem $\Delta^3 x(t) = 0$ satisfying the same boundary conditions (3). In the following discussion we will need the constants

$$k^{-1} := \sum_{s=t_1}^{t_3-1} G(t_2, s) = \frac{1}{6} (t_2 - t_1 + 1)^{(2)} (3t_3 - 2t_2 - t_1 + 2) \quad (21)$$

and

$$m^{-1} := u(h) \sum_{s=t_2-h}^{t_2+h} G(t_2, s) \quad (22)$$

$$= \frac{u(h)}{6} \left[ (t_2 - t_1 + 1)^{(2)} (t_2 - t_1 + 3h + 5) - (t_2 - t_1 - h + 2)^{(3)} \right].$$

Then the growth restrictions on $f$ which will yield the existence of positive and multiple solutions are as follows:

(C1) There exists a $p > 0$ such that $f(t, x) \leq kp$ for $t \in [t_1, t_3 - 1]$ and $0 \leq x \leq p$.

(C2) There exists a $q > 0$ such that $f(t, x) \geq mx$ for $t \in [t_2 - h, t_2 + h]$ and $qu(h) \leq x \leq q$, for $h \in (0, t_3 - t_2 - 1)$.

Many of the following proofs use techniques employed in [16].

Theorem 4. Suppose there exist positive numbers $p \neq q$ such that condition (C1) is satisfied with respect to $p$ and condition (C2) is satisfied with respect to $q$. Then (2), (3) has a positive solution $x$ such that $\|x\|$ lies between $p$ and $q$. 
Proof. This time we take the cone to be

\[ P := \{ x \in B : x(t) \geq u(h)\|x\|, t \in [t_2 - h, t_2 + h] \}, \]

where \( u \) is given in (13). For \( t \in [t_2 - h, t_2 + h] \) and \( x \in P \),

\[
Ax(t) = \sum_{s=t_1}^{t_2-1} G(t, s)f(s, x(s + 1)) \\
\geq g(t) \sum_{s=t_1}^{t_2-1} G(t_2, s)f(s, x(s + 1)) \\
= g(t)\|Ax\| \\
\geq g(t_2 + h)\|Ax\| \\
= u(h)\|Ax\|,
\]

so that \( A(P) \subset P \).

We may assume \( 0 < p < q \) without loss of generality. Define open balls

\[ \Omega_p = \{ x \in B : \|x\| < p \}, \]

and

\[ \Omega_q = \{ x \in B : \|x\| < q \}. \]

Then \( 0 \in \Omega_p \subset \Omega_q \). For \( x \in P \cap \partial \Omega_p \) so that \( \|x\| = p \), we have

\[
\|Ax\| = \sum_{s=t_1}^{t_2-1} G(t_2, s)f(s, x(s + 1)) \\
\leq kp \sum_{s=t_1}^{t_2-1} G(t_2, s) \\
= p \\
= \|x\|
\]

using (C1) and (21). Thus, \( \|Ax\| \leq \|x\| \) for \( x \in P \cap \partial \Omega_p \).

Similarly, let \( x \in P \cap \partial \Omega_q \), so that \( \|x\| = q \). Then

\[
\min_{s \in [t_2 - h, t_2 + h]} x(s + 1) \geq \|x\|u(h),
\]

since \( g(t_2 + h + 1) < g(t_2 + h) \leq g(t_2 - h) \) for all \( h \in (0, t_3 - t_2 - 1) \), \( g \) as in (10), \( u(h) \) as in (13). As a result, \( qu(h) \leq x(s + 1) \leq q \) for \( s \in [t_2 - h, t_2 + h] \), and we
have

\[ ||Ax|| = \sum_{s=t_1}^{t_3-1} G(t_2, s)f(s, x(s+1)) \]
\[ \geq \sum_{s=t_2-h}^{t_2+h} G(t_2, s)f(s, x(s+1)) \]
\[ \geq m \sum_{s=t_2-h}^{t_2+h} G(t_2, s)x(s+1) \]
\[ \geq q \]
\[ = ||x|| \]

by (C2) and (22). Consequently, \( ||Ax|| \geq ||x|| \) for \( x \in \mathcal{P} \cap \partial \Omega_q \). By Theorem 1, \( A \) has a fixed point \( x \in \mathcal{P} \cap (\overline{\Omega}_q \setminus \Omega_p) \), which is a positive solution of (2), (3) such that \( p \leq ||x|| \leq q \). \( \square \)

Define for \( t \in [t_1, t_3 - 1] \)

\[ f_0(t) := \lim_{x \to 0^+} \frac{f(t, x)}{x}, \quad f_\infty(t) := \lim_{x \to \infty} \frac{f(t, x)}{x}. \] (23)

**Corollary 1.** The boundary value problem (2), (3) has a positive solution provided either

(C3) \( f_0(t) < k \) for \( t \in [t_1, t_3 - 1] \) and \( f_\infty(t) > \frac{m}{u(h)} \) for \( t \in [t_2 - h, t_2 + h] \), or

(C4) \( f_0(t) > \frac{m}{u(h)} \) for \( t \in [t_2 - h, t_2 + h] \) and \( f_\infty(t) < k \) for \( t \in [t_1, t_3 - 1] \),

where \( u(h) \) as in (13), \( k \) as in (21), \( m \) as in (22), \( f_0 \) and \( f_\infty \) as in (23).

**Proof.** First assume (C3) holds. Then, there exist sufficiently small \( p > 0 \) and sufficiently large \( q > 0 \) such that

\[ \frac{f(t, x)}{x} \leq k, \quad t \in [t_1, t_3 - 1], \quad 0 < x \leq p \]

and

\[ \frac{f(t, x)}{x} \geq \frac{m}{u(h)}, \quad t \in [t_2 - h, t_2 + h], \quad x \geq qu(h). \]

Then

\[ f(t, x) \leq kx \leq kp, \quad t \in [t_1, t_3 - 1], \quad 0 \leq x \leq p, \]

and

\[ f(t, x) \geq \frac{m}{u(h)} x \geq mq, \quad t \in [t_2 - h, t_2 + h], \quad qu(h) \leq x \leq q. \]
In particular, both \((C_1)\) and \((C_2)\) hold, so that by Theorem 4, (2), (3) has a positive solution.

Next assume \((C_4)\) holds. Then there exist \(0 < p < q\) so that

\[
\frac{f(t,x)}{x} \geq \frac{m}{u(h)}, \ t \in [t_2-h,t_2+h], \ 0 < x \leq p,
\]

\[
\frac{f(t,x)}{x} \leq k, \ t \in [t_1,t_3-1], \ x \geq q.
\]

From (24) we have

\[
f(t,x) \geq \frac{m}{u(h)} x \geq mp, \ t \in [t_2-h,t_2+h], \ pu(h) \leq x \leq p,
\]
satisfying \((C_2)\) with respect to \(p\).

Now consider (25); we wish to show that \((C_1)\) is satisfied. To that end, consider the two cases: \((i)\) \(f(t,x)\) is bounded, or \((ii)\) \(f(t,x)\) is unbounded.

Case \((i)\): Suppose there exists \(N > 0\) such that \(f(t,x) \leq N\), for \(t \in [t_1,t_3-1]\) and \(0 \leq x < \infty\). By (25), there is an \(r \geq \max\{q, \frac{N}{k}\}\) such that \(f(t,x) \leq N \leq kr\) for \(t \in [t_1,t_3-1]\) and \(0 \leq x \leq r\). Thus \((C_1)\) is satisfied with respect to \(r\).

Case \((ii)\): If \(f\) is unbounded, there exist \(t_0 \in [t_1,t_3-1]\) and \(r^* \geq q\) such that \(f(t,x) \leq f(t_0,r^*)\) for \(t \in [t_1,t_3-1]\) and \(0 \leq x \leq r^*\). Then, \(f(t,x) \leq f(t_0,r^*) \leq kr^*\) for \(t \in [t_1,t_3-1]\) and \(0 \leq x \leq r^*\), whence \((C_1)\) is satisfied with respect to \(r^*\).

Thus in both cases \((i)\) and \((ii)\), condition \((C_1)\) is satisfied, and Theorem 4 yields the conclusion.

\[\square\]

4. Existence of Two or More Positive Solutions

In this section, we show that any number [16] of positive solutions of (2), (3) can be obtained when appropriate combinations of assumptions like \((C_1)\), \((C_2)\), \((C_3)\), and \((C_4)\) are imposed on \(f\). We begin the pattern by establishing the existence of at least two positive solutions.

**Theorem 5.** The boundary value problem (2), (3) has at least two positive solutions, \(x_1\) and \(x_2\), if \((C_1)\) is satisfied for some \(p > 0\) and, in addition, both

\[
f_0(t) > \frac{m}{u(h)}, \ t \in [t_2-h,t_2+h], \text{ and } f_\infty(t) > \frac{m}{u(h)}, \ t \in [t_2-h,t_2+h].
\]

Moreover, \(0 < ||x_1|| < p < ||x_2||\).
Proof. Somewhat along the lines of the proof of Corollary 1, there exist $0 < p_1 < p < p_2$ for which

$$f(t, x) \geq mp_1, \quad t \in [t_2 - h, t_2 + h], \quad p_1 u(h) \leq x \leq p_1,$$

and

$$f(t, x) \geq mp_2, \quad t \in [t_2 - h, t_2 + h], \quad p_2 u(h) \leq x \leq p_2.$$

By Theorem 4, there exist solutions $x_1$ and $x_2$ of (2), (3) satisfying $0 < p_1 < ||x|| < p < ||x_2|| < p_2$. □

In a completely analogous manner, the next result is also obtained.

**Theorem 6.** The boundary value problem (2), (3) has at least two positive solutions, $x_1$ and $x_2$, if $(C_2)$ is satisfied for some $q > 0$ and, in addition, both

$$f_0(t) < k, \quad t \in [t_1, t_3 - 1], \quad \text{and} \quad f_\infty(t) < k, \quad t \in [t_1, t_3 - 1]. \quad (27)$$

Moreover, $0 < ||x_1|| < q < ||x_2||$.

To understand the way in which an arbitrary number of positive solutions are obtained, we state an existence result for at least three positive solutions.

**Theorem 7.** [16] Suppose condition $(C_3)$ (or respectively, condition $(C_4)$), is satisfied, and suppose there exist $0 < p_1 < p_2$ such that $(C_1)$ holds with respect to $p = p_2$ (respectively, with respect to $p = p_1$), and $(C_2)$ holds with respect to $q = p_1$ (respectively, with respect to $q = p_2$). Then, the boundary value problem (2), (3) has at least three positive solutions, $x_1$, $x_2$, and $x_3$ satisfying $0 < ||x_1|| < p_1 < ||x_2|| < p_2 < ||x_3||$.

We now state sufficient conditions under which there are $n$ positive solutions of (2), (3) for any $n \in \mathbb{N}$; the theorems are in terms of whether $n$ is odd or even.

**Theorem 8.** [16] Pick any $j \in \mathbb{N}$ and set $n = 2j + 1$. Suppose condition $(C_3)$ (or respectively, condition $(C_4)$) is satisfied, and suppose there exist $0 < p_1 < \cdots < p_{n-1}$ such that $(C_2)$ (respectively, $(C_1)$) holds with respect to $p_{2i-1}$, $1 \leq i \leq j$, and $(C_1)$ (respectively, $(C_2)$) holds with respect to $p_{2i}$, $1 \leq i \leq j$. Then, the boundary value problem (2), (3) has at least $n$ positive solutions $x_1, x_2, \ldots, x_n$ satisfying $0 < ||x_1|| < p_1 < ||x_2|| < p_2 < \cdots < ||x_{n-1}|| < p_{n-1} < ||x_n||$. 

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Theorem 9. [16] For any $j \in \mathbb{N}$ set $n = 2j$. Suppose $(26)$ (or respectively, $(27)$), is satisfied, and suppose there exist $0 < p_1 < \cdots < p_{n-1}$ such that $(C1)$ (respectively, $(C2)$) holds with respect to $p_{2i-1}$, $1 \leq i \leq j$, and $(C2)$ holds (respectively, $(C1)$ holds), with respect to $p_{2i}$, $1 \leq i \leq j-1$. Then, the boundary value problem $(2), (3)$ has at least $n$ positive solutions, $x_1, x_2, \ldots, x_n$ satisfying $0 < ||x_1|| < p_1 < ||x_2|| < p_2 < \cdots < ||x_{n-1}|| < p_{n-1} < ||x_n||$.

References


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