Eigenvalue intervals for a two-point boundary value problem on a measure chain

Douglas R. Anderson

Department of Mathematics and Computer Science, Concordia College, Moorhead, MN 56562, USA

Received 20 August 2000; received in revised form 6 December 2000

Abstract

We study the existence of eigenvalue intervals for the second-order differential equation on a measure chain, \( x^{\Delta\Delta}(t) + \lambda p(t) f(x^{\Delta}(t)) = 0, \ t \in [t_1, t_2] \), satisfying the boundary conditions \( x(t_1) - \beta x^{\Delta}(t_1) = 0 \) and \( \gamma x(\sigma(t_2)) + \delta x^{\Delta}(\sigma(t_2)) = 0 \), where \( f \) is a positive function and \( p \) a nonnegative function that is allowed to vanish on some subintervals of \([t_1, \sigma(t_2)]\) of the measure chain. The methods involve applications of a fixed point theorem for operators on a cone in a Banach space. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 34B99; 39A10; 39A99

Keywords: Fixed point theorems; Green’s function

1. Introduction

One goal as the result of Hilger’s [18] initial paper introducing measure chains has been the unification of the continuous and discrete calculus, and then extending those results to differential equations on time scales. Some other early papers in this area include Agarwal and Bohner [1], Aulbach and Hilger [6] and Erbe and Hilger [12].

One particular area receiving current attention is the question of obtaining optimal eigenvalue intervals of boundary value problems for ordinary differential equations, as well as for finite difference equations. Many of these works have used Krasnoselskii [21] fixed point theorems to obtain intervals based on positive solutions inside a cone. A few papers along these lines are in [3–5, 9,11,16,17,19,20].

Naturally many of these methods carry over when seeking eigenintervals of boundary value problems for differential equations on measure chains; see [2,7,8,13–15], for example.
In this paper, we are concerned with the existence of eigenvalues for a second-order differential equation on a measure chain satisfying Sturm–Liouville-like boundary conditions. Section 2 introduces this boundary value problem.

2. Eigenvalue intervals

Consider the second-order conjugate boundary value problem

\[-x^{ΔΔ}(t) = λ p(t)f(x^σ(t)),\]

\[αx(t_1) − βx^{Δ}(t_1) = 0,\]

\[γx(t_2) + δx^{Δ}(σ(t_2)) = 0,\]

where \(t_1, t_2 \in \mathbb{T},\) a measure chain, with \(t_1 < t_2.\) The Green’s function [13] for the related homogeneous problem \(-x^{ΔΔ}(t) = 0\) with boundary conditions (2) is given by

\[G(t,s) = \begin{cases} \frac{1}{d} \{α(t-t_1) + β\} \{γ(σ(t_2) - σ(s)) + δ\}, & t \leq s, \\ \frac{1}{d} \{α(σ(s) - t_1) + β\} \{γ(σ(t_2) - t) + δ\}, & σ(s) \leq t, \end{cases}\]

where \(α, β, γ, δ ≥ 0\) and \(d := γβ + αδ + γγ(σ(t_2) - t_1) > 0.\)

For the rest of this section we have the assumptions [9,20]

(A1) \(p(t)\) is a nonnegative, right-dense continuous function defined on \([t_1, σ(t_2)]\) satisfying

\[0 < \int_{t_1}^{σ(t_2)} G(σ(s), s)p(s) Δs < ∞,\]

where, using (3),

\[G(σ(s), s) = \frac{1}{d} \{α(σ(s) - t_1) + β\} \{γ(σ(t_2) - σ(s)) + δ\}.\]

(A2) \(f : [0, ∞) → [0, ∞)\) is continuous such that both

\[f_0 := \lim_{x→0^+} \frac{f(x)}{x}\] and \(f_∞ := \lim_{x→∞} \frac{f(x)}{x}\)

exist.

Lemma 1. For all \(s ∈ [t_1, t_2]\) and \(t ∈ [t_1, σ(t_2)]\),

\[g(t)G(σ(s), s) ≤ G(t, s) ≤ G(σ(s), s),\]

where

\[g(t) := \min \left\{ \frac{α(t-t_1) + β}{α(σ(t_2) - t_1) + β}, \frac{γ(σ(t_2) - t) + δ}{γ(σ(t_2) - σ(t_1)) + δ} \right\}.\]
Proof. We have from [13] that
\[
\frac{G(t,s)}{G(\sigma(s),s)} = \begin{cases} 
\frac{\alpha(t - t_1) + \beta}{\alpha(\sigma(s) - t_1) + \beta}, & t \leq s, \\
\frac{\gamma(\sigma(t_2) - t) + \delta}{\gamma(\sigma(t_2) - \sigma(s)) + \delta}, & \sigma(s) \leq t,
\end{cases}
\]
which implies the result. \(\square\)

To establish eigenvalue intervals we will employ the following fixed point theorem due to Krasnoselskii [21], which can also be found in the book in [10].

**Theorem 2.** Let \(E\) be a Banach space, \(K \subseteq E\) be a cone, and suppose that \(\Omega_1, \Omega_2\) are open subsets of \(E\) with \(0 \in \Omega_1\) and \(\bar{\Omega}_1 \subseteq \Omega_2\). Suppose further that \(A: K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K\) is a completely continuous operator such that either
(i) \(\|Au\| \leq \|u\|, \ u \in K \cap \partial \Omega_1\) and \(\|Au\| \geq \|u\|, \ u \in K \cap \partial \Omega_2\), or
(ii) \(\|Au\| \geq \|u\|, \ u \in K \cap \partial \Omega_1\) and \(\|Au\| \leq \|u\|, \ u \in K \cap \partial \Omega_2\)
holds. Then \(A\) has a fixed point in \(K \cap (\bar{\Omega}_2 \setminus \Omega_1)\).

Let \(B\) denote the Banach space \(C_{rd}[t_1, \sigma(t_2)]\) with the norm \(\|x\| = \sup_{t \in [t_1, \sigma(t_2)]} |x(t)|\). Define the cone \(\mathcal{P} \subset B\) by
\[
\mathcal{P} = \{x \in B: x(t) \geq g(t)\|x\|, t \in [t_1, \sigma(t_2)]\},
\]
where \(g\) is given in (6). Let \(t_1 < \xi < \omega < \sigma(t_2)\) be chosen from \(\mathbb{T}\) such that
\[
\int_{\xi}^{\omega} G(\sigma(s),s) p(s) \Delta s > 0, \tag{7}
\]
as \(p\) is a nonnegative function, this allows \(p\) to vanish on some subintervals. For ease of notation in the following discussion, set
\[
k := \min_{t \in [\xi, \sigma(\omega)]} g(t) \tag{8}
\]
for \(g\) as in (6). Moreover, let \(K\) and \(\tau \in [t_1, \sigma(t_2)]\) be defined by
\[
K := g(\tau) = \max_{t \in [t_1, \sigma(t_2)]} g(t). \tag{9}
\]

**Theorem 3.** Suppose (A1) and (A2) hold. Then for each \(\lambda\) satisfying
\[
\frac{1}{\int_{t_1}^{\sigma(t_2)} G(\sigma(s),s) p(s) \Delta s} < \lambda < \frac{1}{\int_{t_1}^{\sigma(t_2)} G(\sigma(s),s) p(s) \Delta s}, \tag{10}
\]
there exists at least one solution of (1) and (2) in \(\mathcal{P}\), for \(\xi, \omega\) as in (7), \(k\) as in (8) and \(K\) as in (9).
Proof. Let $k$ be as in (8), $K$ as in (9), $\lambda$ as in (10), and let $\varepsilon > 0$ be such that
\[
\frac{1}{(f_\infty - \varepsilon)kK} \int_{\tau}^{\sigma} G(\sigma(s), s) p(s) \Delta s \leq \lambda \leq \frac{1}{(f_0 + \varepsilon)\int_{t_1}^{\sigma(t_2)} G(\sigma(s), s) p(s) \Delta s}.
\] (11)

Since $x(t)$ is a solution of (1) and (2) if and only if
\[
x(t) = \lambda \int_{t_1}^{\sigma(t_2)} G(t, s) p(s) f(x^\sigma(s)) \Delta s, \quad t \in [t_1, \sigma(t_2)],
\]
we define the operator $T : \mathcal{P} \rightarrow \mathcal{B}$ by
\[
(Tx)(t) := \lambda \int_{t_1}^{\sigma(t_2)} G(t, s) p(s) f(x^\sigma(s)) \Delta s, \quad x \in \mathcal{P}.
\] (12)

We seek a fixed point of $T$ in $\mathcal{P}$ by establishing the hypotheses of Theorem 2. First, if $x \in \mathcal{P}$ then by (5) we have
\[
(Tx)(t) = \lambda \int_{t_1}^{\sigma(t_2)} G(t, s) p(s) f(x^\sigma(s)) \Delta s
\]
\[
\leq \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s) p(s) f(x^\sigma(s)) \Delta s,
\]
so that
\[
(Tx)(t) = \lambda \int_{t_1}^{\sigma(t_2)} G(t, s) p(s) f(x^\sigma(s)) \Delta s
\]
\[
\geq \lambda g(t) \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s) p(s) f(x^\sigma(s)) \Delta s
\]
\[
\geq g(t) \|Tx\|.
\]

Therefore $T : \mathcal{P} \rightarrow \mathcal{P}$. Moreover, $T$ is completely continuous by a typical application of the Ascoli–Arzelà Theorem.

Now consider $f_0$. There exists an $H_1 > 0$ such that $f(x) \leq (f_0 + \varepsilon)x$ for $0 < x \leq H_1$ by the definition of $f_0$. Pick $x \in \mathcal{P}$ with $\|x\| = H_1$. Using (5) we have
\[
(Tx)(t) = \lambda \int_{t_1}^{\sigma(t_2)} G(t, s) p(s) f(x^\sigma(s)) \Delta s
\]
\[
\leq \lambda (f_0 + \varepsilon)\|x\| \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s) p(s) \Delta s
\]
\[
\leq \|x\|
\]
from the right-hand side of (11). As a result, $\|Tx\| \leq \|x\|$. Thus, take
\[
\Omega_1 := \{x \in \mathcal{B} : \|x\| < H_1\}
\]
so that $\|Tx\| \leq \|x\|$ for $x \in \mathcal{P} \cap \partial \Omega_1$. 
Next consider $f_\infty$. Again by definition there exists an $H_2 > H_1$ such that $f(x) \geq (f_\infty - \varepsilon)x$ for $x \geq H_2$. If $x \in \mathcal{P}$ with $\|x\| = H_2$, then for $t \in [\xi, \sigma(\omega)]$, where $\xi, \omega$ is as in (7) and $k$ as in (8), we have

$$x(t) \geq g(t)\|x\| = g(t)H_2 \geq kH_2. \quad (13)$$

Define $\Omega_2 := \{x \in \mathcal{P}: \|x\| < H_2\}$; using (13) for $\tau$ as in (9), we get

$$(Tx)(\tau) = \lambda \int_{1}^{\sigma(\tau)} G(\tau, s) p(s) f(x^\sigma(s)) \Delta s$$

$$\geq \lambda \int_{\xi}^{\sigma(\tau)} G(\tau, s) p(s) f(x^\sigma(s)) \Delta s$$

$$\geq \lambda g(\tau)(f_\infty - \varepsilon) \int_{\xi}^{\sigma(\tau)} G(\sigma(s), s) p(s)x^\sigma(s) \Delta s$$

$$\geq \lambda (f_\infty - \varepsilon)kKH_2 \int_{\xi}^{\sigma(\tau)} G(\sigma(s), s) p(s) \Delta s$$

$$\geq H_2$$

$$= \|x\|,$$

where the penultimate line follows from the left-hand side of (11). Hence, we have shown that

$$\|Tx\| \geq \|x\|, \quad x \in \mathcal{P} \cap \partial \Omega_2.$$

An application of Theorem 2 validates the conclusion of the theorem. □

**Theorem 4.** Suppose (A1) and (A2) hold. Then for each $\lambda$ satisfying

$$\frac{1}{f_0 k K \int_{\xi}^{\sigma(\tau)} G(\sigma(s), s) p(s) \Delta s} < \lambda \leq \frac{1}{\int_{1}^{\sigma(\tau)} G(\sigma(s), s) p(s) \Delta s} \quad (14)$$

there exists at least one solution of (1) and (2) in $\mathcal{P}$.

**Proof.** Let $\lambda$ be as in (14) and let $\eta > 0$ be such that

$$\frac{1}{(f_0 - \eta) k K \int_{\xi}^{\sigma(\tau)} G(\sigma(s), s) p(s) \Delta s} \leq \lambda \leq \frac{1}{(f_\infty + \eta) \int_{1}^{\sigma(\tau)} G(\sigma(s), s) p(s) \Delta s}. \quad (15)$$

Let $T$ be the completely continuous, cone-preserving operator defined in (12). We seek a fixed point of $T$ in $\mathcal{P}$ by establishing the hypotheses of Theorem 2.

First, consider $f_0$. There exists an $H_1 > 0$ such that $f(x) \geq (f_0 - \eta)x$ for $0 < x \leq H_1$ by the definition of $f_0$. Pick $x \in \mathcal{P}$ with $\|x\| = H_1$. For $t \in [\xi, \sigma(\omega)]$, where $\xi, \omega$ is as in (7) and $k$ as in (8), we have

$$x(t) \geq g(t)\|x\| = g(t)H_1 \geq kH_1. \quad (16)$$
Using the left-hand side of (15) and (16) and $\tau$ from (9), we get

\[(Tx)(\tau) = \lambda \int_{t_1}^{\sigma(t_2)} G(\tau, s) p(s) f(x^\sigma(s)) \Delta s \]

\[\geq \lambda \int_{\xi}^{\sigma(t_2)} G(\tau, s) p(s) f(x^\sigma(s)) \Delta s \]

\[\geq \lambda (f_0 - \eta) \int_{\xi}^{\sigma(t_2)} G(\sigma(s), s) p(s) x^\sigma(s) \Delta s \]

\[\geq \lambda (f_0 - \eta) H_1 kK \int_{\xi}^{\sigma(t_2)} G(\sigma(s), s) p(s) \Delta s \]

\[\geq H_1 \]

\[= \|x\|.

Therefore $\|Tx\| \geq \|x\|$. This prompts us to define

$\Omega_1 := \{x \in \mathcal{B}: \|x\| < H_1\}$,

whereby our work above confirms

$\|Tx\| \geq \|x\|, \quad x \in \mathcal{P} \cap \bar{\mathcal{B}} \Omega_1$.

Next consider $f_\infty$. Again by definition there exists an $\tilde{H}_2 > H_1$ such that $f(x) \leq (f_\infty + \eta)x$ for $x \geq \tilde{H}_2$. If $f$ is bounded, there exists $M > 0$ with $f(x) \leq M$ for all $x \in (0, \infty)$. Let

$H_2 := \max \left\{ 2\tilde{H}_2, \lambda M \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s) p(s) \Delta s \right\}$.

If $x \in \mathcal{P}$ with $\|x\| = H_2$, then we have

\[(Tx)(t) = \lambda \int_{t_1}^{\sigma(t_2)} G(t, s) p(s) f(x^\sigma(s)) \Delta s \]

\[\leq \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s) p(s) f(x^\sigma(s)) \Delta s \]

\[\leq \lambda M \int_{t_1}^{\sigma(t_2)} G(\sigma(s), s) p(s) \Delta s \]

\[\leq H_2 \]

\[= \|x\|.

As a result, $\|Tx\| \leq \|x\|$. Thus, take

$\Omega_2 := \{x \in \mathcal{B}: \|x\| < H_2\}$
so that \( \|Tx\| \leq \|x\| \) for \( x \in \mathcal{P} \cap \partial \Omega_2 \). If \( f \) is unbounded, take \( H_2 := \max\{2H_1, \tilde{H}_2\} \) such that \( f(x) \leq f(H_2) \) for \( 0 < x \leq H_2 \). If \( x \in \mathcal{P} \) with \( \|x\| = H_2 \), then we have

\[
(Tx)(t) = \lambda \int_{t_1}^{\sigma(t_2)} G(t,s) p(s) f(x^c(s)) \Delta s \\
\leq \lambda \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s) p(s) f(H_2) \Delta s \\
\leq \lambda (f_{\infty} + \eta) H_2 \int_{t_1}^{\sigma(t_2)} G(\sigma(s),s) p(s) \Delta s \\
\leq H_2 \\
= \|x\|,
\]

where the penultimate line follows from the left-hand side of (15). Hence, we have shown that

\[
\|Tx\| \leq \|x\|, \quad x \in \mathcal{P} \cap \partial \Omega_2
\]

if we take

\[
\Omega_2 := \{x \in \mathcal{P} : \|x\| < H_2\}.
\]

Once again an application of Theorem 2 validates the conclusion of the theorem. \( \square \)

References


