HYERS-ULAM STABILITY OF HIGHER-ORDER CAUCHY-EULER
DYNAMIC EQUATIONS ON TIME SCALES

DOUGLAS R. ANDERSON

Department of Mathematics, Concordia College, Moorhead, MN 56562 USA

Dedicated to Professor John R. Graef on the occasion of his 70th birthday.

ABSTRACT. We extend a recent result on third and fourth-order Cauchy-Euler equations by establishing the Hyers-Ulam stability of higher-order linear non-homogeneous Cauchy-Euler dynamic equations on time scales. That is, if an approximate solution of a higher-order Cauchy-Euler equation exists, then there exists an exact solution to that dynamic equation that is close to the approximate one. We generalize this to all higher-order linear non-homogeneous factored dynamic equations with variable coefficients.

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1. INTRODUCTION

Stan Ulam [27] posed the following problem concerning the stability of functional equations: give conditions in order for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by Hyers [10], who proved that the Cauchy equation is stable in Banach spaces, and the result of Hyers was generalized by Rassias [24]. Obloza [19] appears to be the first author who investigated the Hyers-Ulam stability of a differential equation.

Since then there has been a significant amount of interest in Hyers-Ulam stability, especially in relation to ordinary differential equations, for example see [7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 25, 28]. Also of interest are many of the articles in a special issue guest edited by Rassias [23], dealing with Ulam, Hyers-Ulam, and Hyers-Ulam-Rassias stability in various contexts. Also see Popa et al [5, 20, 21, 22]. András and Mészáros [2] recently used an operator approach to show the stability of linear dynamic equations on time scales with constant coefficients, as well as for certain integral equations. Tunç and Biçer [26] proved the Hyers-Ulam stability of third and fourth-order Cauchy-Euler differential equations. Anderson et al [1, Corollary 2.6] proved the following concerning second-order non-homogeneous Cauchy-Euler equations on time scales:
Theorem 1.1 (Cauchy-Euler Equation). Let $\lambda_1, \lambda_2 \in \mathbb{R}$ (or $\lambda_2 = \overline{\lambda_1}$, the complex conjugate) be such that

$$t + \lambda_k \mu(t) \neq 0, \quad k = 1, 2$$

for all $t \in [a, \sigma(b)]_\mathbb{T}$, where $a \in \mathbb{T}$ satisfies $a > 0$. Then the Cauchy-Euler equation

$$x^{\Delta k}(t) + \frac{1 - \lambda_1 - \lambda_2}{\sigma(t)} x^{\Delta}(t) + \frac{\lambda_1 \lambda_2}{\tau \sigma(t)} x(t) = f(t), \quad t \in [a, b]_\mathbb{T}$$

has Hyers-Ulam stability on $[a, b]_\mathbb{T}$. To wit, if there exists $y \in C_{\text{id}}^{\Delta k}[a, b]_\mathbb{T}$ that satisfies

$$\left| y^{\Delta k}(t) + \frac{1 - \lambda_1 - \lambda_2}{\sigma(t)} y^{\Delta}(t) + \frac{\lambda_1 \lambda_2}{\tau \sigma(t)} y(t) - f(t) \right| \leq \varepsilon$$

for $t \in [a, b]_\mathbb{T}$, then there exists a solution $u \in C_{\text{id}}^{\Delta k}[a, b]_\mathbb{T}$ of (1.1) given by

$$u(t) = e_{\lambda_1}(t, \tau_2) y(\tau_2) + \int_{\tau_2}^{t} e_{\lambda_1}(t, \sigma(s)) w(s) \Delta s, \quad \text{any} \quad \tau_2 \in [a, \sigma^2(b)]_\mathbb{T},$$

where for any $\tau_1 \in [a, \sigma(b)]_\mathbb{T}$ the function $w$ is given by

$$w(s) = e_{\lambda_2-1}(s, \tau_1) \left[ y^{\Delta}(\tau_1) - \frac{\lambda_1}{\tau_1} y(\tau_1) \right] + \int_{\tau_1}^{s} e_{\lambda_2-1}(s, \sigma(\zeta)) f(\zeta) \Delta \zeta,$$

such that $|y - u| \leq K\varepsilon$ on $[a, \sigma^2(b)]_\mathbb{T}$ for some constant $K > 0$.

The motivation for this work is to extend Theorem 1.1 to the general $n$th-order Cauchy-Euler dynamic equation, and thus extend the results in [26] as well, with an approach different from [2]. We will show the stability in the sense of Hyers and Ulam of the equation

$$\sum_{k=0}^{n} \alpha_k M_k y(t) = f(t),$$

where

$$M_0 y(t) := y(t), \quad M_{k+1} y(t) := \varphi(t) (M_k y)^{\Delta}(t), \quad k = 0, 1, \ldots, n - 1.$$

This is essentially [4, (2.14)] if $\varphi(t) = t$ and $f(t) = 0$. In the last section we will analyze the $n$th-order factored equation with differential operators $D$ and $I$, where $Dy = y^{\Delta}$ and $Iy = y$, of the form

$$\prod_{k=1}^{n} (\varphi_k D - \psi_k I) y(t) = f(t), \quad t \in [a, b]_\mathbb{T},$$

for right-dense continuous functions $\varphi_k$ and $\psi_k$, a more general dynamic equation with variable coefficients than the Cauchy-Euler equation. Throughout this work we assume the reader has a working knowledge of time scales as can be found in Bohner and Peterson [3, 4], originally introduced by Hilger [9].
2. HYERS-ULAM STABILITY FOR HIGHER-ORDER CAUCHY-EULER DYNAMIC EQUATIONS

In this section we establish the Hyers-Ulam stability of the higher-order non-homogeneous Cauchy-Euler dynamic equation on time scales of the form

\begin{equation}
\sum_{k=0}^{n} \alpha_k M_k y(t) = f(t),
\end{equation}

where

\[ M_0 y(t) := y(t), \quad M_{k+1} y(t) := \varphi(t) (M_k y)^\Delta(t), \quad k = 0, 1, \ldots, n - 1 \]

for given constants \( \alpha_k \in \mathbb{R} \) with \( \alpha_n \equiv 1 \), and for functions \( \varphi, f \in C_{rd}[a, b]_{\mathbb{T}} \), using the following definition.

**Definition 2.1** (Hyers-Ulam stability). Let \( \varphi, f \in C_{rd}[a, b]_{\mathbb{T}} \) and \( n \in \mathbb{N} \). If whenever \( M_k x \in C_{rd}^\Delta[a, b]_{\mathbb{T}} \) satisfies

\[
\left| \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t) \right| \leq \varepsilon, \quad t \in [a, b]_{\mathbb{T}}
\]

there exists a solution \( u \) of (2.1) with \( M_k u \in C_{rd}^\Delta[a, b]_{\mathbb{T}} \) for \( k = 0, 1, \ldots, n - 1 \) such that \( |x - u| \leq K \varepsilon \) on \([a, \sigma^n(b)]_{\mathbb{T}}\) for some constant \( K > 0 \), then (2.1) has Hyers-Ulam stability \([a, b]_{\mathbb{T}}\).

**Remark 2.2.** Before proving the Hyers-Ulam stability of (2.1) we will need the following lemma, which allows us to factor (2.1) using the elementary symmetric polynomials [6] in the \( n \) symbols \( \rho_1, \ldots, \rho_n \) given by

\[
s_1^n = s_1(\rho_1, \ldots, \rho_n) = \sum \rho_i,
\]
\[
s_2^n = s_2(\rho_1, \ldots, \rho_n) = \sum_{i<j} \rho_i \rho_j,
\]
\[
s_3^n = s_3(\rho_1, \ldots, \rho_n) = \sum_{i<j<k} \rho_i \rho_j \rho_k,
\]
\[
s_4^n = s_4(\rho_1, \ldots, \rho_n) = \sum_{i<j<k<\ell} \rho_i \rho_j \rho_k \rho_\ell,
\]
\[\vdots\]
\[
s_t^n = s_t(\rho_1, \ldots, \rho_n) = \sum_{i_1<i_2<\cdots<i_t} \rho_{i_1} \rho_{i_2} \cdots \rho_{i_t},
\]
\[\vdots\]
\[
s_n^n = s_n(\rho_1, \ldots, \rho_n) = \rho_1 \rho_2 \rho_3 \cdots \rho_n.
\]
In general, we let \( s_i^n \) represent the \( i \)th elementary symmetric polynomial on \( j \) symbols. Then, given the \( \alpha_k \) in (2.1), introduce the characteristic values \( \lambda_k \in \mathbb{C} \) via the elementary symmetric polynomial \( s_i^n \) on the \( n \) symbols \(-\lambda_1, \ldots, -\lambda_n\), where \( \alpha_n = s_0 \equiv 1 \) and

\[
\alpha_k = s_{n-k}^n = s_{n-k}(-\lambda_1, \ldots, -\lambda_n) = \sum_{i_1 < i_2 < \cdots < i_{n-k}} (-1)^{n-k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-k}}. \tag{2.2}
\]

**Lemma 2.3** (Factorization). Given \( y, \varphi \in C_{rd}[a,b]_T \) and \( \alpha_k \in \mathbb{R} \) with \( \alpha_n \equiv 1 \), let \( M_k y \in C_{rd}[a,b]_T \), where \( M_0 y(t) := y(t) \) and \( M_{k+1} y(t) := \varphi(t) (M_k y)^\Delta(t) \) for \( k = 0, 1, \ldots, n - 1 \). Then we have the factorization

\[
\sum_{k=0}^{n} \alpha_k M_k y(t) = \prod_{k=1}^{n} (\varphi D - \lambda_k I) y(t), \quad n \in \mathbb{N}, \tag{2.3}
\]

where the differential operator \( D \) is defined via \( D x = x^\Delta \) for \( x \in C_{rd}[a,b]_T \), and \( I \) is the identity operator.

**Proof.** We proceed by mathematical induction on \( n \in \mathbb{N} \), utilizing the substitution defined in (2.2). For \( n = 1 \),

\[
\sum_{k=0}^{n} \alpha_k M_k y(t) = \alpha_0 M_0 y(t) + \alpha_1 M_1 y(t) = s_1(-\lambda_1) y(t) + 1 \cdot \varphi(t) y^\Delta(t) = (\varphi D - \lambda_1 I) y(t)
\]

and the result holds. Assume (2.3) holds for \( n \geq 1 \). Then we have \( \alpha_{n+1} \equiv 1 \) and

\[
\sum_{k=0}^{n+1} \alpha_k M_k y(t) = \alpha_0 y(t) + \sum_{k=1}^{n} \alpha_k M_k y(t) + M_{n+1} y(t)
\]

\[
= s_{n+1}^{n+1} y(t) + \sum_{k=1}^{n} s_{n+1-k}^{n+1} M_k y(t) + \varphi(t) (M_n y)^\Delta(t)
\]

\[
= -\lambda_{n+1} s_{n}^{n} y(t) + \sum_{k=1}^{n} \left( s_{n+1-k}^{n} - \lambda_{n+1} s_{n-k}^{n} \right) M_k y(t) + \varphi(t) D (M_n y)(t)
\]

\[
= -\lambda_{n+1} \left[ s_{n}^{n} y(t) + \sum_{k=1}^{n} s_{n-k}^{n} M_k y(t) \right] + \sum_{k=1}^{n} s_{n+1-k}^{n} M_k y(t)
\]

\[
+ \varphi(t) D (M_n y)(t)
\]

\[
= -\lambda_{n+1} \sum_{k=0}^{n} s_{n-k}^{n} M_k y(t) + \varphi(t) D \left( \sum_{k=1}^{n} s_{n+1-k}^{n} M_{k-1} y(t) + M_n y \right)(t)
\]

\[
= -\lambda_{n+1} \sum_{k=0}^{n} s_{n-k}^{n} M_k y(t) + \varphi(t) D \left( \sum_{k=0}^{n-1} s_{n-k}^{n} M_k y(t) + M_n y \right)(t)
\]

\[
= -\lambda_{n+1} \sum_{k=0}^{n} s_{n-k}^{n} M_k y(t) + \varphi(t) D \sum_{k=0}^{n} s_{n-k}^{n} M_k y(t)
\]

\[
= -\lambda_{n+1} \sum_{k=0}^{n} s_{n-k}^{n} M_k y(t) + \varphi(t) D \sum_{k=0}^{n} s_{n-k}^{n} M_k y(t)
\]

\[
= \sum_{k=0}^{n} \alpha_k M_k y(t),
\]

as required.
\[
= (\varphi(t)D - \lambda_{n+1}I) \sum_{k=0}^{n} s_{n-k}^n M_k y(t)
\]
\[
= (\varphi(t)D - \lambda_{n+1}I) \sum_{k=0}^{n} \alpha_k M_k y(t)
\]
\[
= (\varphi(t)D - \lambda_{n+1}I) \prod_{k=1}^{n} (\varphi D - \lambda_k I) y(t)
\]

and the proof is complete. \hfill \square

**Theorem 2.4** (Hyers-Ulam Stability). Given \( y, \varphi, f \in C_{rd}[a, b]_T \) with \(|\varphi| \geq A > 0\) for some constant \( A \), and \( \alpha_k \in \mathbb{R} \) with \( \alpha_n \equiv 1 \), consider \( M_k y \in C_{rd}[a, b]_T \) for \( k = 0, \ldots, n - 1 \). Using the \( \lambda_k \) from the factorization in Lemma 2.3, assume

\[
(2.4) \quad \varphi(t) + \lambda_k \mu(t) \neq 0, \quad k = 1, 2, \ldots, n
\]

for all \( t \in [a, \sigma^{n-1}(b)]_T \). Then \( (2.1) \) has Hyers-Ulam stability on \([a, b]_T\).

**Proof.** Let \( \varepsilon > 0 \) be given, and suppose there is a function \( x \), with \( M_k x \in C_{rd}[a, b]_T \), that satisfies

\[
\left| \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t) \right| \leq \varepsilon, \quad t \in [a, b]_T.
\]

We will show there exists a solution \( u \) of \((2.1)\) with \( M_k u \in C_{rd}[a, b]_T \) for \( k = 0, 1, \ldots, n - 1 \) such that \( |x - u| \leq K\varepsilon \) on \([a, \sigma^n(b)]_T\) for some constant \( K > 0 \).

To this end, set

\[
\begin{align*}
g_1 &= \varphi x^\Delta - \lambda_1 x = (\varphi D - \lambda_1 I) x \\
g_2 &= \varphi g_1^\Delta - \lambda_2 g_1 = (\varphi D - \lambda_2 I) g_1 \\
& \vdots \\
g_k &= \varphi g_{k-1}^\Delta - \lambda_k g_{k-1} = (\varphi D - \lambda_k I) g_{k-1} \\
& \vdots \\
g_n &= \varphi g_{n-1}^\Delta - \lambda_n g_{n-1} = (\varphi D - \lambda_n I) g_{n-1}.
\end{align*}
\]

This implies by Lemma 2.3 that

\[
g_n(t) - f(t) = \sum_{k=0}^{n} \alpha_k M_k x(t) - f(t),
\]

so that

\[
|g_n(t) - f(t)| \leq \varepsilon, \quad t \in [a, b]_T.
\]

By the construction of \( g_n \) we have \( |\varphi g_{n-1}^\Delta - \lambda_n g_{n-1} - f| \leq \varepsilon \), that is

\[
\left| \frac{\varphi}{\varphi} g_{n-1}^\Delta - \lambda_n g_{n-1} - f \right| \leq \frac{\varepsilon}{|\varphi|} \leq \frac{\varepsilon}{A}.
\]
By [1, Lemma 2.3] and (2.4) there exists a solution \( w_1 \in C^\Delta_{id}[a, b]_T \) of
\[
(2.5) \quad w^\Delta(t) - \frac{\lambda_n}{\varphi(t)} w(t) - \frac{f(t)}{\varphi(t)} = 0, \quad \text{or} \quad \varphi(t) w^\Delta(t) - \lambda_n w(t) - f(t) = 0,
\]
t \in [a, b]_T, where \( w_1 \) is given by
\[
w_1(t) = e^{\frac{\lambda_n}{\varphi}(t, \tau_1)} g_{n-1}(\tau_1) + \int_{\tau_1}^t e^{\frac{\lambda_n}{\varphi}(t, \sigma(s))} \frac{f(s)}{\varphi(s)} \Delta s, \quad \text{any} \quad \tau_1 \in [a, \sigma(b)]_T,
\]
and there exists an \( L_1 > 0 \) such that
\[
|g_{n-1}(t) - w_1(t)| \leq L_1 \varepsilon/A, \quad t \in [a, \sigma(b)]_T.
\]
Since \( g_{n-1} = \varphi g^\Delta_{n-2} - \lambda_{n-1} g_{n-2} \), we have that
\[
|\varphi g^\Delta_{n-2} - \lambda_{n-1} g_{n-2} - w_1(t)| \leq L_1 \varepsilon/A, \quad t \in [a, \sigma(b)]_T.
\]
Again we apply [1, Lemma 2.3] to see that there exists a solution \( w_2 \in C^\Delta_{id}[a, \sigma(b)]_T \) of
\[
(2.6) \quad w^\Delta(t) - \frac{\lambda_{n-1}}{\varphi(t)} w(t) - \frac{w_1(t)}{\varphi(t)} = 0, \quad \text{or} \quad \varphi(t) w^\Delta(t) - \lambda_{n-1} w(t) - w_1(t) = 0,
\]
t \in [a, \sigma(b)]_T, where \( w_2 \) is given by
\[
w_2(t) = e^{\frac{\lambda_{n-1}}{\varphi}(t, \tau_2)} g_{n-2}(\tau_2) + \int_{\tau_2}^t e^{\frac{\lambda_{n-1}}{\varphi}(t, \sigma(s))} \frac{w_1(s)}{\varphi(s)} \Delta s, \quad \text{any} \quad \tau_2 \in [a, \sigma^2(b)]_T,
\]
and there exists an \( L_2 > 0 \) such that
\[
|g_{n-2}(t) - w_2(t)| \leq L_2 L_1 \varepsilon/A^2, \quad t \in [a, \sigma^2(b)]_T.
\]
Continuing in this manner, we see that for \( k = 1, 2, \ldots, n-1 \) there exists a solution \( w_k \in C^\Delta_{id}[a, \sigma^{k-1}(b)]_T \) of
\[
(2.7) \quad w^\Delta(t) - \frac{\lambda_{n-k+1}}{\varphi(t)} w(t) - \frac{w_{k-1}(t)}{\varphi(t)} = 0, \quad \text{or} \quad \varphi(t) w^\Delta(t) - \lambda_{n-k+1} w(t) - w_{k-1}(t) = 0,
\]
t \in [a, \sigma^{k-1}(b)]_T, where \( w_k \) is given by
\[
w_k(t) = e^{\frac{\lambda_{n-k+1}}{\varphi}(t, \tau_k)} g_{n-k}(\tau_k) + \int_{\tau_k}^t e^{\frac{\lambda_{n-k+1}}{\varphi}(t, \sigma(s))} \frac{w_{k-1}(s)}{\varphi(s)} \Delta s, \quad \text{any} \quad \tau_k \in [a, \sigma^k(b)]_T,
\]
and there exists an \( L_k > 0 \) such that
\[
|g_{n-k}(t) - w_k(t)| \leq \prod_{j=1}^{k} L_j \varepsilon/A^k, \quad t \in [a, \sigma^k(b)]_T.
\]
In particular, for \( k = n-1 \),
\[
|g_1(t) - w_{n-1}(t)| \leq \prod_{j=1}^{n-1} L_j \varepsilon/A^{n-1}, \quad t \in [a, \sigma^{n-1}(b)]_T.
\]
implies by the definition of \( g_1 \) that

\[
| x^\Delta(t) - \frac{\lambda_1}{\varphi(t)} x(t) - \frac{w_{n-1}(t)}{\varphi(t)} | \leq \prod_{j=1}^{n-1} L_j \varepsilon / A^n, \quad t \in [a, \sigma^{n-1}(b)]_T.
\]

Thus there exists a solution \( w_n \in C^\Delta_{rd}[a, \sigma^{n-1}(b)]_T \) of

\[
w^\Delta(t) - \frac{\lambda_1}{\varphi(t)} w(t) - \frac{w_{n-1}(t)}{\varphi(t)} = 0, \quad \text{or} \quad \varphi(t) w^\Delta(t) - \lambda_1 w(t) - w_{n-1}(t) = 0,
\]

\( t \in [a, \sigma^{n-1}(b)]_T \), where \( w_n \) is given by

\[
w_n(t) = e_{\lambda_1}(t, \tau_n) x(\tau_n) + \int_{\tau_n}^t e_{\lambda_1}(t, s) \frac{w_{n-1}(s)}{\varphi(s)} \Delta s, \quad \text{any} \quad \tau_n \in [a, \sigma^n(b)]_T,
\]

and there exists an \( L_n > 0 \) such that

\[
| x(t) - w_n(t) | \leq K \varepsilon := \prod_{j=1}^{n} L_j \varepsilon / A^n, \quad t \in [a, \sigma^n(b)]_T.
\]

By construction,

\[
(\varphi D - \lambda_1 I) w_n(t) = w_{n-1}(t)
\]

\[
\prod_{k=1}^{2} (\varphi D - \lambda_k I) w_n(t) = (\varphi D - \lambda_2 I) w_{n-1}(t) = w_{n-2}(t)
\]

\[
\vdots
\]

\[
\prod_{k=1}^{n} (\varphi D - \lambda_k I) w_n(t) = (\varphi D - \lambda_n I) w_1(t) \overset{\text{(2.5)}}{=} f(t)
\]

on \([a, \sigma^{n-1}(b)]_T\), so that \( u = w_n \) is a solution of (2.1), with \( u \in C^\Delta_{rd}[a, \sigma^{n-1}(b)]_T \) and \( | x(t) - w_n(t) | \leq K \varepsilon \) for \( t \in [a, \sigma^n(b)]_T \) by (2.8). Moreover, using (2.7) and (2.6), we have an iterative formula for this solution \( u = w_n \) in terms of the function \( x \) given at the beginning of the proof. \( \square \)

### 3. EXAMPLE

Letting \( Dy = y^5 \) and \( I \) be the identity operator, consider the non-homogeneous fifth-order Cauchy-Euler dynamic equation

\[
\left[ (tD)^5 + 15(tD)^4 + 85(tD)^3 + 225(tD)^2 + 274tD + 120I \right] y(t) = f(t)
\]

for some right-dense continuous function \( f \), on \([a, b]_T\); in factored form it is

\[
(tD + 5I)(tD + 4I)(tD + 3I)(tD + 2I)(tD + I)y(t) = f(t).
\]

If \( T = \mathbb{R} \), this is equivalent to the non-homogeneous fifth-order Cauchy-Euler differential equation

\[
t^5 y^{(5)} + 25t^4 y^{(4)} + 200t^3 y''' + 600t^2 y'' + 600ty' + 120y = f(t),
\]

By Theorem 2.4 we have that (3.1) has Hyers-Ulam stability.
4. HIGHER-ORDER LINEAR NON-HOMOGENEOUS FACTORED
DYNAMIC EQUATIONS WITH VARIABLE COEFFICIENTS

Generalizing away from higher-order Cauchy-Euler equations, we consider the
following higher-order linear non-homogeneous factored dynamic equations with vari-
able coefficients given by

\[(4.1)\quad \prod_{k=1}^{n} (\varphi_k D - \psi_k I) y(t) = f(t), \quad t \in [a,b)_T,\]

where \(\varphi_k, \psi_k, f \in C_{rd}[a,b)_T\) for \(k = 1, 2, \ldots, n\), \(Dy(t) = y^\Delta(t)\), \(I\) is the identity
operator, and \(|\varphi_k(t)| \geq A > 0\) for all \(t \in [a,b)_T\), for some constant \(A > 0\). Here we
allow for \(b = \infty\) for those time scales that are unbounded above. Before our main
result in this section we need the following lemma.

**Lemma 4.1.** Let \(\varphi, \psi, f \in C_{rd}[a,b)_T\) with \(|\varphi(t)| \geq A > 0\) for some constant \(A\), and assume

\[(4.2)\quad \varphi(t) + \mu(t) \psi(t) \neq 0 \quad \text{and} \quad \int_{a}^{t} \left| e_{\varphi}(t, \sigma(\tau)) \frac{1}{\varphi(\tau)} \right| \Delta \tau < L\]

for all \(t \in [a,b)_T\), for some constant \(0 < L < \infty\). Then the first-order dynamic
equation

\[ (\varphi D - \psi I) y - f = 0 \]

has Hyers-Ulam stability on \([a,b)_T\).

**Proof.** Suppose there exists a function \(x\) such that

\[ |(\varphi D - \psi I) x(t) - f(t)| \leq \varepsilon \]

for some \(\varepsilon > 0\), for all \(t \in [a,b)_T\). Set

\[ q(t) = (\varphi D - \psi I) x(t) - f(t), \quad t \in [a,b)_T. \]

Clearly \(|q(t)| \leq \varepsilon\) for all \(t \in [a,b)_T\), and we can solve for \(x\) to obtain

\[ x(t) = e_{\varphi}(t, a)x(a) + \int_{a}^{t} e_{\varphi}(t, \sigma(\tau)) \frac{q(\tau) + f(\tau)}{\varphi(\tau)} \Delta \tau. \]

Let \(y\) be the unique solution of the initial-value problem

\[ (\varphi D - \psi I) y(t) - f(t) = 0, \quad y(a) = x(a). \]

Then \(y\) is given by

\[ y(t) = e_{\varphi}(t, a)x(a) + \int_{a}^{t} e_{\varphi}(t, \sigma(\tau)) \frac{f(\tau)}{\varphi(\tau)} \Delta \tau, \]

and

\[ |y(t) - x(t)| = \left| \int_{a}^{t} e_{\varphi}(t, \sigma(\tau)) \frac{q(\tau)}{\varphi(\tau)} \Delta \tau \right| \leq \varepsilon \int_{a}^{t} \left| e_{\varphi}(t, \sigma(\tau)) \frac{1}{\varphi(\tau)} \right| \Delta \tau \leq L \varepsilon \]

by condition (4.2). \(\square\)
Theorem 4.3

nym function.

for all integers 

\( t, \sigma = 1, \ldots, n \) for all integers 

\( t, \tau \), and so clearly converges on \([3, \infty)\), where \( \Gamma \) is the gamma function.

Remark 4.2. The convergence condition on the integral in (4.2) is essentially the same as S1 and S2 in [2], and can be met for various functions. For example, if \( \mathbb{T} = \mathbb{R} \), 

\( \varphi(w) = w^2, \psi(w) = \sin w, \) and \( a = 1 \), then 

\[
\int_{a}^{t} \left| e_{\varphi(t, \sigma)} \frac{1}{\varphi(t)} \right| \Delta \tau = \int_{1}^{t} e_{\sin \varphi(w)} \frac{1}{\varphi(t)} d\tau \in \left[ 1 - e^{-1+\frac{1}{4}}, -1 + e^{1+\frac{1}{4}} \right]
\]

for all \( t \geq 1 \), and so clearly converges on \([1, \infty)\). If \( \mathbb{T} = \mathbb{N} \), \( \varphi(w) = w, \psi(w) = -5/2, \) and \( a = 3 \), then 

\[
\int_{a}^{t} \left| e_{\varphi(t, \sigma)} \frac{1}{\varphi(t)} \right| \Delta \tau = \sum_{\tau = 3}^{t-1} \frac{\Gamma(1+\tau)\Gamma(t-5/2)}{\tau \Gamma(t)\Gamma(t-3/2)} = \frac{2}{5} - \frac{4 \Gamma(t-5/2)}{5 \sqrt{\pi} \Gamma(t)} \in [0, 2/5)
\]

for all integers \( t \geq 3 \), and so clearly converges on \([3, \infty)\), where \( \Gamma \) is the gamma function.

Theorem 4.3 (Hyers-Ulam Stability). Given \( \varphi, \psi, f \in C_{rd}[a, b]_{\mathbb{T}} \) with \( |\varphi_k(t)| \geq A > 0 \) for some constant \( A \), assume

\[
\varphi_k(t) + \mu(t)\psi_k(t) \neq 0 \quad \text{and} \quad \int_{a}^{t} \left| e_{\varphi_k(t)} \frac{1}{\varphi(t)} \right| \Delta \tau < L_{k-1}
\]

for \( k = 1, 2, \ldots, n \) and for all \( t \in [a, b]_{\mathbb{T}} \), where \( 0 < L_{k-1} < \infty \) is some constant. Then (4.1) has Hyers-Ulam stability on \([a, b]_{\mathbb{T}}\).

Proof. Suppose there exists a function \( x \) such that

\[
\left| \prod_{k=1}^{n} (\varphi_k D - \psi_k I) x(t) - f(t) \right| \leq \varepsilon
\]

for some \( \varepsilon > 0 \), for all \( t \in [a, b]_{\mathbb{T}} \). Define the new functions \( x_0 := x, y_n := f, \) and

\[
x_k := (\varphi_k D - \psi_k I) x_{k-1}, \quad k = 1, \ldots, n.
\]

Then

\[
x_k(t) = \varphi_k(t)x_{k-1}(t) - \psi_k(t)x_{k-1}(t),
\]

that can be solved to yield

\[
x_{k-1}(t) = e_{\psi_k(t)}(t, a)x_{k-1}(a) + \int_{a}^{t} e_{\psi_k(t)}(t, \sigma) \frac{x_k(\tau)}{\varphi_k(\tau)} \Delta \tau \]

for \( k = 1, \ldots, n \). Note that

\[
|x_n - y_n|(t) = |x_n - f|(t) \leq \varepsilon,
\]

so

\[
|\varphi_n x_{n-1}^\Delta - \psi_n x_{n-1} - y_n| = |\varphi_n x_{n-1}^\Delta - \psi_n x_{n-1} - f| \leq \varepsilon.
\]

Hyers-Ulam stability of this first-order equation by Lemma 4.1 implies there exists a function \( y_{n-1} \) such that

\[
x_{n-1} - y_{n-1} \leq L_{n-1} \varepsilon
\]
and

\[ \varphi_n(t) y_{n-1}^\Delta(t) - \psi_n(t) y_{n-1}(t) = y_n(t) = f(t), \]

for some constant \( L_{n-1} > 0 \), where \( y_{n-1} \) is given by

\[ y_{n-1}(t) = e_{\varphi_{n-1}}(t, a) y_{n-1}(a) + \int_a^t e_{\varphi_{n-1}}(t, \sigma(\tau)) \frac{y_n(\tau)}{\varphi_n(\tau)} \Delta \tau. \]

Then

\[ |\varphi_{n-1} x_{n-2}^\Delta - \psi_{n-1} x_{n-2} - y_{n-1}| \leq L_{n-1} \epsilon, \]

so again Hyers-Ulam stability of the first-order equation implies there exists a function \( y_{n-2} \) such that

\[ |x_{n-2} - y_{n-2}| \leq L_{n-2} L_{n-1} \epsilon \]

and

\[ \varphi_{n-1}(t) y_{n-2}^\Delta(t) - \psi_{n-1}(t) y_{n-2}(t) = y_{n-1}(t), \]

for some constant \( L_{n-2} > 0 \), where \( y_{n-2} \) is given by

\[ y_{n-2}(t) = e_{\varphi_{n-1}}(t, a) y_{n-2}(a) + \int_a^t e_{\varphi_{n-1}}(t, \sigma(\tau)) \frac{y_{n-1}(\tau)}{\varphi_{n-1}(\tau)} \Delta \tau. \]

Continuing in this way, we obtain a function \( y_0 \) such that

\[ |x_0 - y_0| = |x - y_0| \leq \varepsilon \prod_{j=0}^{n-1} L_j \]

and

\[ \varphi_1(t) y_0^\Delta(t) - \psi_1(t) y_0(t) = y_1(t), \]

for some constant \( L_0 > 0 \), where \( y_0 \) is given by

\[ y_0(t) = e_{\varphi_1}(t, a) y_0(a) + \int_a^t e_{\varphi_1}(t, \sigma(\tau)) \frac{y_1(\tau)}{\varphi_1(\tau)} \Delta \tau. \]

Note that by construction of \( y_0 \) and generally \( y_k \), we have

\[ \prod_{k=1}^n (\varphi_k D - \psi_k I) y_0(t) = \prod_{k=2}^n (\varphi_k D - \psi_k I) y_1(t) \]

\[ = \prod_{k=3}^n (\varphi_k D - \psi_k I) y_2(t) \]

\[ = \cdots \]

\[ = (\varphi_n D - \psi_n I) y_{n-1}(t) = y_n(t) = f(t), \]

making \( y_0 \) a solution of (4.1). This fact, together with inequality (4.5), shows that (4.1) has Hyers-Ulam stability. \( \square \)
REFERENCES


