Multiple Positive Solutions to a Third-Order Discrete Focal Boundary Value Problem

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Abstract—We are concerned with the discrete focal boundary value problem
\[ \Delta^3 x(t-k) = f(x(t)), \]
\[ x(a) = x(t_2) = \Delta^2 x(b + 1) = 0. \]
Under various assumptions on f and the integers a, t2, and b we prove the existence of three positive solutions of this boundary value problem. To prove our results, we will apply a generalization of the Leggett-Williams fixed-point theorem. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Difference equations, Green's function, Fixed points.

1. PRELIMINARIES

In this paper, we are concerned with the existence of three positive solutions to the third-order boundary value problem

\[ -\Delta^3 x(t-k) + f(x(t)) = 0, \quad \text{for all } t \in [a+k, b+k], \]
\[ x(a) = \Delta x(t_2) = \Delta^2 x(b + 1) = 0, \]
where \( f : \mathbb{R} \to \mathbb{R} \) is continuous, \( f \) is nonnegative for \( x \geq 0 \), and \( k \in \{1, 2\} \). A solution of (1),(2) is nonnegative on \([a, b+3]\), nondecreasing on \([a, t_2]\), and nonincreasing on \([t_2, b+3]\). In [1], Anderson, Avery and Peterson imposed conditions on \( f \) to yield at least three positive solutions to (1),(2) applying the Leggett-Williams fixed-point theorem. Henderson and Thompson [2] and Avery and Henderson [3] used the symmetry of Green's function to improve known results for the existence of solutions to a second-order conjugate problem. Although the associated Green's function to the boundary value problem (1),(2) is not symmetric, we were able to employ the techniques of [2,3] to improve the known results of [1]. For a third-order continuous case, see [4].

The literature on positive solutions to boundary value problems is extensive. The recent book by Agarwal, Wong and O'Regan [5] gives a good overview for much of the work which has been
done and the methods used. In this paper, we will assume the reader has an understanding of Green's functions and their applications. Books on the subject include [6,7]. In the remainder of this section, we will state the generalization of the Leggett-Williams fixed-point theorem [8], which will be used to prove our main result, and provide some background results and definitions.

**DEFINITION 1.** Let $E$ be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by

$x \leq y$, if and only if $y - x \in P$.

**DEFINITION 2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**DEFINITION 3.** A map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ if

$\alpha : P \rightarrow [0, \infty)$

is continuous and

$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$,

for all $x, y \in P$ and $t \in [0,1]$. Similarly, we say the map $\beta$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ if

$\beta : P \rightarrow [0, \infty)$

is continuous and

$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$

for all $x, y \in P$ and $t \in [0,1]$.

Let $\gamma, \beta, \theta$ be nonnegative continuous convex functionals on $P$ and $\alpha, \psi$ be nonnegative continuous concave functionals on $P$; then for nonnegative real numbers $h, a, b, d, c$, we define the following convex sets:

$P(\gamma, c) = \{x \in P : \gamma(x) < c\}$,
$P(\gamma, \alpha, a, c) = \{x \in P : a \leq \alpha(x), \gamma(x) \leq c\}$,
$Q(\gamma, \beta, d, c) = \{x \in P : \beta(x) \leq d, \gamma(x) < c\}$,
$P(\gamma, \theta, a, b, c) = \{x \in P : a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\}$,
$Q(\gamma, \beta, \psi, h, d, c) = \{x \in P : h \leq \psi(x), \beta(x) \leq d, \gamma(x) \leq c\}$.

The following fixed-point theorem is a generalization of the Leggett-Williams fixed-point theorem due to Avery [8].

**THEOREM 4.** Let $P$ be a cone in a real Banach space $E$; $c$, $M$ be positive numbers; $\alpha, \psi$ be nonnegative continuous concave functionals on $P$; and $\gamma, \beta, \theta$ be nonnegative continuous convex functionals on $P$ with

$\alpha(x) \leq \beta(x)$ and $\|x\| \leq M\gamma(x),$

for all $x \in P(\gamma, c)$. Suppose

$$A : P(\gamma, c) \rightarrow P(\gamma, c).$$
is completely continuous and there exist nonnegative numbers \( h, d, a, b \) with \( 0 < d < a \) such that

(i) \( \{ x \in P(\gamma, \theta, \alpha, a, b, c) : \alpha(x) > a \} \neq \emptyset \) and \( \alpha(Ax) > a \) for \( x \in P(\gamma, \theta, \alpha, a, b, c) \);

(ii) \( \{ x \in Q(\gamma, \beta, \psi, h, d, c) : \beta(x) < d \} \neq \emptyset \) and \( \beta(Ax) < d \) for \( x \in Q(\gamma, \beta, \psi, h, d, c) \);

(iii) \( \alpha(Ax) > a \) for \( x \in P(\gamma, \alpha, a, c) \) with \( \theta(Ax) > b \);

(iv) \( \beta(Ax) < d \) for \( x \in Q(\gamma, \beta, d, c) \) with \( \psi(Ax) < h \).

Then \( A \) has at least three fixed points \( x_1, x_2, x_3 \in P(\gamma, c) \) such that

\[ \beta(x_1) < d, \quad a < \alpha(x_2), \quad \text{and} \quad d < \beta(x_3), \quad \text{with} \quad \alpha(x_3) < a. \]

2. INTRODUCTION TO THE THIRD-ORDER BVP

We are concerned with proving the existence of three positive solutions of the third-order nonlinear focal boundary value problem

\[ -\Delta^3 x(t - k) + f(x(t)) = 0, \quad \text{for all} \quad t \in [a + k, b + k], \]

with boundary conditions

\[ x(a) = x(b + 1) = 0, \]

where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous, \( f \) is nonnegative for \( x \geq 0 \), and \( k \in \{1, 2\} \). The solutions of (1),(2) are the fixed points of the operator \( A \) defined by

\[ Ax(t) = \sum_{s=a-k}^{b+k} G(t,s)f(x(s)), \]

where \( G(t,s) \) is the Green's function for the operator \( L \) defined by

\[ Lx(t) = \Delta^3 x(t - k) \]

with boundary conditions

\[ x(a) = \Delta x(t_2) = \Delta^2 x(b + 1) = 0. \]

Define [6] the factorial function by

\[ t(n) = t(t-1) \cdots (t-n+1) \]

for \( t \) a real number and \( n \) a positive integer. Then the Green's function is given [9] by

\[ G(t,s) = \begin{cases} 
    u_1(t,s), & t < s - k + 3, \\
    v_1(s), & t \geq s - k + 1,
\end{cases} \]

\[ s \in [a + k, t_2 + k - 1] : \]

\[ \begin{cases} 
    u_2(t), & t < s - k + 3, \\
    v_2(t,s), & t \geq s - k + 1,
\end{cases} \]

\[ s \in [t_2 + k - 1, b + k] : \]

where

\[ u_1(t,s) = \begin{vmatrix} 0 & t-a & \frac{1}{2}(t-a)^{(2)} \\ -(s-k+1-t_2) & 1 & t_2-a \\ 1 & 0 & 1 \end{vmatrix}, \]

\[ v_1(s) = \frac{1}{2} (s-k-a+2)^{(2)}, \]

\[ u_2(t) = \begin{vmatrix} t-a & \frac{1}{2}(t-a)^{(2)} \\ 1 & t_2-a \end{vmatrix}, \]

\[ v_2(t,s) = u_2(t) + \frac{1}{2} (t-s+k-1)^{(2)}. \]

By [9], if \( t_2 - a \geq b - t_2 + 2 \),

\[ G(t_2,s) \geq G(t,s) > 0, \]

for \( t \in (a, b + 3], s \in [a + k, b + k] \). Throughout this paper, we assume that

\[ t_2 - a > b - t_2 + 2. \]
LEMMA 5. Let \( t_2 \in \{0, 1, \ldots, b + 3 - t_2\} \). Then
\[
G(t_2 - h, s) \leq G(t_2 + h, s),
\]
for all \( s \in [a - k, b - k] \).

PROOF. For \( s \in [a-k, t_2 - k - 1 - h] \), we know that \( t_2 + h \geq t_2 - h \geq s - k + 1 \). As a result,
\[
G(t_2 + h, s) - G(t_2 - h, s) = 0.
\]

If \( s \in [t_2 + k - 1 - h, t_2 + k - 1] \), it follows that \( t_2 - h \leq s - k + 1 \leq t_2 + h \), so that
\[
G(t_2 - h, s) - G(t_2 - h, s) = \frac{1}{2}(s - k - a + 2)^2 - \frac{1}{2}(t_2 - h - a)(2s - 2k - a - t_2 + h + 3)
\]
\[
= \frac{1}{2}(s - k - a + 2)^2
\]
\[
- \frac{1}{2}(t_2 - h - a)(s - k - a - 2) + (s - k - t_2 + h - 1)
\]
\[
= \frac{1}{2}(s - k - t_2 + h - 2)^2
\]
\[
\geq 0.
\]

for these \( s \). Next let \( s \in [t_2 + k - 1, t_2 + k - 1 + h] \). Then
\[
G(t_2 + h, s) - G(t_2 - h, s) = \frac{1}{2}(t_2 + h - a)(t_2 - h - a + 1) + \frac{1}{2}(t_2 + h - s + k - 1)^2
\]
\[
- \frac{1}{2}(t_2 - h - a)(t_2 - h - a + 1)
\]
\[
= h + \frac{1}{2}(t_2 + h - s + k - 1)^2
\]
\[
\geq 0.
\]

Finally, consider \( s \in [t_2 + k - 1 + h, b + k] \). Then
\[
G(t_2 + h, s) - G(t_2 - h, s) = \frac{1}{2}[(t_2 - h - a)(t_2 + h - a + 1) - (t_2 + h - a)(t_2 - h - a + 1)]
\]
\[
- h
\]
\[
\geq 0.
\]

3. INEQUALITIES AND EQUALITIES NEEDED IN THE EXISTENCE THEOREMS

Define the Banach space \( E \) by
\[
E = \{ y \mid y : [a, b + 3] \to \mathbb{R}, y(a) = 0 \}
\]
with the supnorm and the cone \( P \) of \( E \) by
\[
P = \left\{ y \in E \mid \begin{array}{l}
y \text{ is nondecreasing on } [a, t_2], \text{ y is nonincreasing on } [t_2, b + 3], \\
y \text{ is nonnegative valued on } [a, b + 3], \\
y(t_2 + h) \geq y(t_2 - h) \text{ for all } h \in [0, b + 3 - t_2], \text{ and} \\
y(t_2 + j) \geq f_3 |y|
\end{array} \right\},
\]
where \( f_3 \) is given in (5).
For integers $h, j, k_1, k_2$ where

\[0 \leq h \leq b + 3 - t_2,\]
\[0 \leq j \leq b + 3 - t_2,\] and
\[1 \leq k_1 \leq k_2 < b + 3 - t_2,\]
define the concave functionals on the cone $P$,

\[\alpha(y) := \min_{t \in [t_2 - t_2, j_2 - k_1]} y(t) = y(t_2 - k_2),\]

and

\[\psi(y) := \min_{t \in [t_2 - h, t_2 + h]} y(t) = y(t_2 - h),\]

and the convex functionals on the cone $P$,

\[\beta(y) := \max_{t \in [t_2, h, t_2 + h]} y(t) = y(t_2),\]
\[\gamma(y) := \max_{t \in [a, t_2 - j], [t_2 + j, b + 3]} y(t) = y(t_2 + j),\] and
\[\theta(y) := \max_{t \in [t_2 - k_2, t_2 - k_1], [k_1, t_2 + k_2]} y(t) = y(t_2 - k_1).\]

We will make use of various properties and constants associated with the Green's function, which include the sums

\[C_1 := \sum_{s = a + k}^{b + k} G(t_2 + j, s) = \frac{1}{24} (t_2 + j - a + 1)^2(3)\]
\[- \frac{1}{8} (t_2 + j - a)(j + a - t_2 - 1)(6b - a - j - 3t_2 + 7),\]

\[C_2 := \frac{t_2}{s - a + k} G(t_2, s) = \frac{1}{6} (t_2 - h + k - a + 2)^2(3),\]

\[C_3 := \sum_{s = t_2 - k_2}^{t_2 + k_2} G(t_2, s) = \frac{1}{2} (t_2 - a + 1)^2(h + k - t_2 - h),\]

\[C_4 := \sum_{s = t_2 - k_2}^{t_2 - k_1} G(t_2, s) = \frac{1}{2} (t_2 - k_2 - a + 1)^2(h - k_2 - k + 1) + \frac{1}{6} (t_2 - a + 2)^2(3)\]
\[- \frac{1}{6} (t_2 - h - k - a + 2)^2(3),\]

\[C_5 := \sum_{s = t_2 - k_2}^{t_2 - k_2} G(t_2 - k_2, s) + \sum_{s = t_2 + k_2}^{t_2 + k_2} G(t_2 - k_2, s)\]
\[= \frac{1}{2} \left( t_2 - k_2 - a \right) \left[ (t_2 + k_2 - a + 1)(k_2 + a - 1) \right.\]
\[\left. + \left( \frac{1}{2} t_2 - k_1 + \frac{1}{2} k_2 - \frac{1}{2} a - k + \frac{5}{2} \right)^2(2) \right.\]
\[\left. - \left( \frac{1}{2} t_2 - k_2 - \frac{1}{2} a - k + \frac{3}{2} \right)^2(2) \right] + \left( \frac{1}{2} t_2 - k_2 - \frac{1}{2} a - 3 \right)^2(3),\]

\[+ \left( \frac{1}{2} t_2 - k_2 - \frac{1}{2} a - 1 \right)^2(2)\]

\[+ \left( \frac{1}{2} t_2 - k_2 - \frac{1}{2} a + \frac{3}{2} \right)^2(2) - \left( \frac{1}{2} t_2 - k_2 - \frac{1}{2} a - k + \frac{3}{2} \right)^2(2),\]

and the constants

\[I_1 := \max_{a + k \leq s \leq b + k} G(t_2, s) = \frac{(t_2 - a + 1)^2(2)}{(t_2 - h - a)(t_2 + h - a + 1)},\]
\[I_2 := \max_{a + k \leq s \leq b + k} G(t_2, k_2, s) = \frac{(t_2 - k_2 - a)(t_2 + k_2 + a + 1)}{(t_2 - a + 1)^2(2)}\]
\[I_3 := \max_{a + k \leq s \leq b + k} G(t_2, j, s) - \frac{(t_2 + j - a)(t_2 + a - j + 1)}{(t_2 - a + 1)^2(2)}.\]
4. THEOREM ON THE EXISTENCE OF THREE POSITIVE SOLUTIONS

In this section, we state and prove a theorem on the existence of three positive solutions to the BVP (1),(2). By a positive solution of the BVP (1),(2), we mean a solution which is in the cone defined in the proof of the following theorem.

**Theorem 6.** Suppose $a'$, $b'$, and $c'$ are nonnegative real numbers with $a' < b' < c'$ such that $f$ satisfies the following conditions:

(i) $f(w) < (a' - c'(C_2 + C_3)/C_1)/C_4$ for all $w \in [a'/I_1, a]$;
(ii) $f(w) > b'/C_5$ for $w \in [b', b'/I_3]$;
(iii) $f(w) \leq c'/C_1$ for $w \in [0, c'/I_3]$.

Then, the discrete third-order boundary value problem (1),(2) has three positive solutions $y_1$, $y_2$, $y_3 \in P(\gamma, c')$.

**Proof.** Define the completely continuous operator $A$ by

$$Ay(t) = \sum_{s=a+k}^{b+k} G(t, s)f(y(s)).$$

We seek fixed points of $A$ which satisfy the conclusion of the theorem. We note first, if $y \in P$, then from properties of $G(t, s)$,

$$A_y(t) \geq 0,$$
$$\Delta A_y(t) \geq 0, \quad \text{for } t \in [a, t_2],$$
$$\Delta A_y(t) \leq 0, \quad \text{for } t \in [t_2, b + 2],$$
$$A_y(t_2 - h) \leq A_y(t_2 + h), \quad \text{for } h \in [0, b + 3 - t_2],$$
$$A_y(t_2 + j) \geq I_5 A_y(t_2);$$

consequently, $A_y \in P$; that is, $A : P \rightarrow P$.

Note that for all $y \in P$,

$$\alpha(y) = y(t_2 - k_2) \leq y(t_2) = \beta(y)$$

and

$$\|y\| \leq \frac{1}{I_5} y(t_2 + j) = \frac{1}{I_5} \gamma(y).$$

If $y \in \overline{P}(\gamma, c')$, then $\|y\| \leq (1/I_3) \gamma(y) \leq c'/I_3$ and by Assumption (iii), we have

$$\gamma(Ay) = \max_{r \in [a, t_2-j] \cup [t_2+j, b+3]} \sum_{s=a+k}^{b+k} G(t, s)f(y(s))$$

$$= \sum_{s=a+k}^{b+k} G(t_2 + j, s)f(y(s))$$

$$\leq \frac{c'}{C_1} \sum_{s=a+k}^{b+k} G(t_2 + j, s)$$

$$= c'.$$

Therefore,

$$A : \overline{P}(\gamma, c') \rightarrow \overline{P}(\gamma, c').$$
It is immediate that

\[ \{ y \in P \left( \gamma, \beta, \alpha, b', \frac{b'}{I_2}, c' \right) : \alpha(y) > b' \} \neq \emptyset \quad \text{and} \]

\[ \{ y \in Q \left( \gamma, \beta, \psi, \frac{a'}{I_1}, a', c' \right) : \beta(y) < a' \} \neq \emptyset. \]

In the following claims, we verify the remaining conditions of the generalized Leggett-Williams fixed-point theorem.

**Claim 1.** If \( y \in Q(\gamma, \beta, a', c') \) with \( \psi(Ay) < a'/I_1 \), then \( \beta(Ay) < a' \).

\[
\beta(Ay) = \max_{t \in [t_2-h, t_2+h]} \sum_{s=a+k}^{b+k} G(t, s)f(y(s))
\]

\[ = \sum_{s=a+k}^{b+k} G(t_2, s)f(y(s)) \]

\[ = \sum_{s=a+k}^{b+k} \frac{G(t_2, s)}{G(t_2-h, s)}G(t_2-h, s)f(y(s)) \]

\[ \leq I_1 \sum_{s=a+k}^{b+k} G(t_2-h, s)f(y(s)) \]

\[ = I_1 \psi(Ay) < a'. \]

**Claim 2.** If \( y \in Q(\gamma, \beta, a'/I_1, a', c') \), then \( \beta(Ay) < a' \).

\[
\beta(Ay) = \max_{t \in [t_2-h, t_2+h]} \sum_{s=a+k}^{b+k} G(t, s)f(y(s))
\]

\[ = \sum_{s=a+k}^{b+k} G(t_2, s)f(y(s)) + \sum_{s=t_2-h}^{t_2+h} G(t_2-h, s)f(y(s)) + \sum_{s=t_2-h+1}^{b+k} G(t_2, s)f(y(s)) \]

\[ < \left( \frac{C_1}{C_1} \right) \sum_{s=a+k}^{b+k} G(t_2, s) - \left( \frac{a'-c'(C_2+C_3)/C_1}{C_1} \right) \sum_{s=t_2-h}^{t_2+h} G(t_2, s) \]

\[ - \left( \frac{C_1}{C_1} \right) (C_2 + C_3) + \left( \frac{a'-c'(C_2+C_3)/C_1}{C_1} \right) C_4 - a'. \]

**Claim 3.** If \( y \in P(\gamma, \alpha, b', c') \) with \( \theta(Ay) > b'/I_2 \), then \( \alpha(Ay) > b' \).

\[
\alpha(Ay) = \min_{t \in [t_2-k_2, t_2-k_2]} \sum_{s=a+k}^{b+k} G(t, s)f(y(s))
\]

\[ = \sum_{s=a+k}^{b+k} G(t_2-k_2, s)f(y(s)) \]

\[ = \sum_{s=a+k}^{b+k} \left( \frac{G(t_2-k_2, s)}{G(t_2-k_1, s)} \right) G(t_2-k_1, s)f(y(s)) \]

\[ > I_2 \sum_{s=a+k}^{b+k} G(t_2-k_1, s)f(y(s)) \]

\[ = I_2 \theta(Ay) > b'. \]
**Claim 4.** If \( y \in P(\gamma, \theta, \alpha, b', b' / T_2, c') \), then \( \alpha(Ay) > b' \).

\[
\alpha(Ay) = \min_{t \in [t_2 - k_2, t_2 - k_2]} \sum_{s = \alpha+k}^{b+k} G(t, s) f(y(s)) \\
\geq \sum_{s = \alpha+k}^{t_2 - k_2} G(t_2 - k_2, s) f(y(s)) + \sum_{s = \alpha+k}^{t_2 + k_2} G(t_2 - k_2, s) f(y(s)) \\
\geq \left( \frac{b'}{C_5} \right) \left( \sum_{s = \alpha+k}^{t_2 - k_2} G(t_2 - k_2, s) + \sum_{s = \alpha+k}^{t_2 + k_2} G(t_2 - k_2, s) \right) \\
= \left( \frac{b'}{C_5} \right) C_5 = b'.
\]

Therefore, the hypotheses of the generalized Leggett-Williams fixed-point theorem are satisfied, and there exist three positive solutions \( y_1, y_2, y_3 \in P(\gamma, c') \) for the third-order discrete focal boundary value problem such that

\[
\alpha(y_1) > b', \\
\beta(y_2) < a', \quad \text{and} \quad \alpha(y_3) < b', \quad \text{with} \quad \beta(y_3) > a'.
\]

**References**