PROJECT 2

HARVESTING NATURAL RESOURCES

You envy these salmon farmers. They don't have to rope or tie their product, they don't have to brand it, and best of all they don't have to smell its wastes. Moreover, they always know how many salmon they will be able to sell. All they have to do is feed their product and apply makeup to it on the way to the market. If only you could have that same kind of security.

You are a fisheries manager for the state of Alaska, and it is your job to help establish regulations for commercial salmon fishing in territorial waters near Glacier Bay, Alaska.* Your problem is that you never know how many fish are out there or which species will dominate in a given year. People who fish salmon for a living depend on the limits you set to put food on their tables and cars in their garages. On the other hand, you have a fishery that could collapse if you set limits that are too high.

National Oceanic and Atmospheric Administration/Department of Commerce

FIGURE 1 Fish trawler near Glacier Bay, Alaska

Commercial fishing exploits a natural and wild resource. In contrast to, say, a cattle farmer who can breed new calves to replenish the herd continually, fishermen taking a wild resource cannot control their supply except by crude harvest limitations. Thus commercial and recreational fishing are regulated in an attempt to limit the harvesting and preserve the resource. The effects of harvesting a renewable natural resource such as fish can be modeled using a modification of the **logistic equation:**

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - H(P). \tag{1}$$

The first term on the right-hand side of the DE in (1) is a model of the population growth with a growth rate $r > 0$ and a carrying capacity (maximum sustainable population) of $K > 0$. See Section 3.2 for a more detailed discussion of this population model. The second term $H(P)$ is the *harvesting* term. You want to look at two different forms of $H(P)$. Each form will correspond to a possible regulatory strategy, and it is your task to understand how these strategies affect the long-term fish population.

You begin by considering what happens in the logistic model without harvesting—that is, when $H(P) = 0$. Let P denote the salmon population in thousands and define

$$\ell(P) = rP\left(1 - \frac{P}{K}\right).$$

PROBLEM 1. Graph $y = \ell(P)$ and use this graph to sketch approximate solution curves of (1) in the tP-plane when $H(P) = 0$. What are the critical points of this autonomous DE? Use phase line analysis—that is, a one-dimensional phase portrait—to give the stability classification of each critical point.

PROBLEM 2. Interpret the solution curves sketched in Problem 1 in terms of the long-time behavior of the salmon population. Begin with the equilibrium solutions of the DE and then interpret the solution curves corresponding to various possible initial conditions.

*Glacier Bay is located in Glacier Bay National Park in the Panhandle of southeast Alaska.

Now that you understand what happens without human intervention, you begin to consider what happens if we start harvesting the salmon. You begin by assuming that $r = 1$ and that the carrying capacity is $K = 1000$. Your objective is to determine a harvest rate for the fishing industry. Thus instead of assigning a number to the harvest rate, you simply assume that the harvesting occurs at a constant rate of h thousand salmon per year. With $H(P) = h$, (1) becomes

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - h, \quad h > 0. \tag{2}$$

The DE in (2) is called the **constant-harvest model.** Your goal is to understand what happens to the salmon population as h increases.

PROBLEM 3 (CD). In (2), let $f_h(P) = rP(1 - P/1000) - h$. Use the **Logistic Harvest Tool** on the *DE Tools* CD to examine $f_h(P)$ for $0 < h < 300$. Describe what happens to the critical points of the DE and the corresponding phase lines. Relate this behavior to the population model. In particular, for each harvesting level h, what initial population levels ultimately lead to extinction, and how does this change with h?

PROBLEM 4. For what value of h is there only one critical point? What does the phase line look like for values of h slightly smaller than and slightly larger than this value? Again, interpret this in terms of the model for the salmon population.

One thing that you need to care about is a dependence on specific numbers in the model. You don't actually know the ocean's carrying capacity K or a species growth rate r. Studies and experiments suggest values for these parameters, but even then they are only estimates. It is very important for you to understand the behavior of the model as a whole. Otherwise, you could make a mistake in parameter value and wipe out the fishery.

PROBLEM 5. Repeat Problem 3 without assigning numerical values to r and K. Use calculus techniques rather than the CD to sketch a graph of $f_h(P)$.

The constant-harvest model (2) that you have examined corresponds to a simple approach to licensing in which fishermen are allowed a constant take, regardless of the time required for that. Another approach is to assume that harvesting is proportional to the population present. In other words, instead of allowing the same number of fish to be harvested each year, you allow only a fraction of the present population to be caught. In this scenario we write $H(P) = \alpha P$, $0 \leq \alpha < 1$, and (1) becomes

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - \alpha P. \tag{3}$$

The differential equation in (3) is called the **proportional harvesting model.**

PROBLEM 6. Compute the critical points of the proportional harvesting model (3) and use phase line analysis to classify their stability. Interpret your results in terms of the salmon population.

PROBLEM 7. Now it's time to prepare your report and suggest a policy to your supervisor. Compare and contrast these two harvesting strategies. Be sure to make note of the strengths and limitations of each model. Brainstorm with your colleagues on other possible harvesting strategies and model these as well. Write a report making a suggestion for a regulatory strategy based on the work you've done.

STILL CURIOUS?

The problems above dealing with constant harvesting provide an example of a **bifurcation.** A bifurcation is essentially a dramatic change in the qualitative structure of the phase line, such as the appearance or disappearance of a critical point. In what follows we look more carefully at bifurcations in an attempt to determine when that might occur.

We say that a critical point of the autonomous DE $x' = f(x)$ is **hyperbolic** if small perturbations of $f(x)$ *do not* change the qualitative structure of the differential equation. Critical points are allowed to move slightly but not vanish, appear out of the blue, or change their stability. If arbitrary small changes *do* change the nature of the system, we say that the critical point is **nonhyperbolic.**

Consider a family of autonomous differential equations $x' = f_a(x)$, where a is a parameter. A bifurcation diagram for a family of DEs is simply a graph that shows the location and stability of the critical points for each parameter a. Consider the example above with $r = 1$, $K = 1000$, and constant harvesting. The critical points satisfy the quadratic equation

$$P\left(1 - \frac{P}{1000}\right) - h = 0. \tag{4}$$

The equations of the top and bottom curves of the parabola shown in Figure 2 are thus obtained from (4) using the quadratic formula. Their stability can be easily determined as well: The top curve is the asymptotically stable critical point, and the bottom curve is the unstable critical point.

PROBLEM 8 (CAS). Find the equations defined by (4). Then use a computer algebra system to plot the graph shown in Figure 2.

Note that for $h < 250$ there are two critical points, and a small change in h does not change either the number or the stability of these critical points. These critical

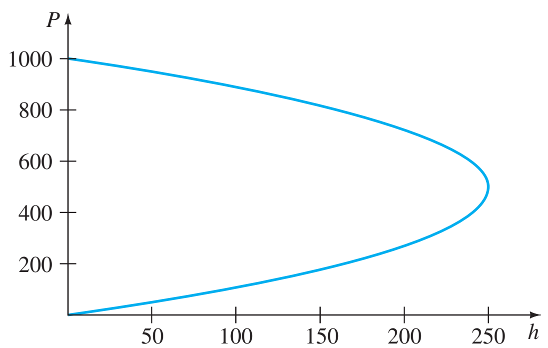

FIGURE 2 Bifurcation diagram for (1) when $H(P)$ is a constant

points are hyperbolic. However, when $h = 250$, there is only one critical point which is nonhyperbolic, since an infinitesimally small change of h leads to a dramatic change in the qualitative structure. In other words, if you move h any distance at all from 250, then the number of critical points changes from one to two or from one to zero.

You see that qualitative changes in the dynamical structure of the family of differential equations only occur at a nonhyperbolic critical point. This change is called a **bifurcation,** and the parameter value at which the bifurcation occurs is called a **bifurcation point.**

Your supervisor does not care about hyperbolic critical points, and she does *not* want to hear about bifurcations. However, she does care about the health of the fishery. For that reason you want to choose limits on your harvest that cause the equilibria to be well to the left of the bifurcation point.

PROBLEM 9. Why do you want to choose a value of h that is to the left of the bifurcation point on the graph?